



## VOLUME MINIMIZATION WITH DISPLACEMENT CONSTRAINTS IN TOPOLOGY OPTIMIZATION OF CONTINUUM STRUCTURES

A. Csébfalvi<sup>\*†</sup>

*University of Pécs, Pécs, Hungary*

### ABSTRACT

In this paper, a displacement-constrained volume-minimizing topology optimization model is present for two-dimensional continuum problems. The new model is a generalization of the displacement-constrained volume-minimizing model developed by Yi and Sui [1] in which the displacement is constrained in the loading point. In the original model the displacement constraint was formulated as an equality relation, which practically means that the number of “interesting points” may be exactly one. The recent model resolves this weakness replacing the equality constraint with an inequality constraint. From engineering point of view it is a very important result because we can replace the inequality constraint with a set of inequality constraints without any difficulty. The other very important fact, that the modified displacement-oriented model can be extended very easily to handle stress-oriented relations, which will be demonstrated in the forthcoming paper. Naturally, the more general theoretical model needs more sophisticated numerical problem handling method. Therefore, we replaced the original “optimality-criteria-like” solution searching process with a standard nonlinear programming approach which is able to handle linear (nonlinear) objectives with linear (nonlinear) equality (inequality) constraints. The efficiency of the new approach is demonstrated by an example investigated by several authors. The presented example with reproducible numerical results as a benchmark problem may be used for testing the quality of exact and heuristic solution procedures to be developed in the future for displacement-constrained volume-minimization problems.

**KEYWORDS:** topology optimization; volume minimization; displacement constraint; ground structure, element density.

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<sup>\*</sup>Corresponding author: University of Pécs, Pécs, Hungary

<sup>†</sup>E-mail address: csebfalvi.witch@gmail.com (A. Csébfalvi)

## 1. INTRODUCTION

The topology optimization subjected to displacement constraints lead to a challenge from computational point of view. Already in 1993, Sankaranarayanan et al. [2] have been presented a pioneer work for truss topology optimization problems with displacement constraints. Compliance minimization has been popular for obtaining optimum truss topologies because it is computationally less demanding than optimization for more general constraints. This advantage of compliance minimization led to the approach of optimizing the topology of the truss for minimum compliance, and then sizing the cross-sectional areas of that truss for the actual constraints. This paper shows that the compliance minimization can occasionally lead to the wrong topology, and that the penalty associated with using the minimum-compliance topology can be substantial. To demonstrate this problem truss topology optimization subject to displacement and stress constraints is performed with a simultaneous analysis and design (SAND) procedure.

Bendsøe and Ben-Tal [3] presented a displacement based method for maximum stiffness truss topology design where ground structure approach is used, and the problem is formulated in terms of displacements and bar areas. This large, non—convex optimization problem can be solved by identifying an equivalent, unconstrained and convex problem in the displacement which can be solved by a non—smooth, steepest algorithm. In this method an explicit solving of the equilibrium equations and the assembly of the global stiffness matrix is applied. A large number of examples have been studied, showing the attractive features of topology design as well as exposing interesting features of optimal topologies.

In 1997, M. Kočvara [4] published a bilevel programming approach for topology optimization with displacement constraints. Minimum-compliance formulation of the truss topology problem with additional linear constraints on the displacements is considered. In the frame of bilevel programming approach the primal goal (upper-level) is to satisfy the displacement constraint as well as possible minimize the gap between the actual and prescribed displacement. Second goal (lower level) is to minimize the compliance still want to find the stiffest structure satisfying the displacement constraints. This approach leads to a nonsmooth optimization problem which is finally solved by a nonsmooth solver.

In a recent paper, Yi and Sui [1] presented a displacement-constrained volume-minimizing model in which the displacement was constrained in the loading point. The authors developed an “optimality-criteria-like” solution searching algorithm to generate the optimal solution. The applied algorithm exploits the fact that the displacement is constrained at only one “interesting point” which is by definition the loading point. Therefore the algorithm is very similar to the algorithm used to minimize the compliance with exactly one equality constraint which defines the allowed volume fraction. In the Yi and Sui [1] model, the displacement constraint is formulated by an equality relation, therefore the number of “interesting points” can be exactly one.

Sui and Yi [5] presented earlier a discussion about choosing an objective function and constraint conditions in structural topology optimization. In order to solve a topology optimization model related to practical engineering problems a weight minimization with a displacement constraint is discussed. The authors developed a 120-line code, written in Matlab, based on some examples calculated using the 99-line code made by Sigmund [6]. An improvement of the previous paper has been suggested by Andreassen et al. [7] using the

99 line code presented by Sigmund [6] as a starting point. The original code has been extended by a density filter, and a considerable improvement in efficiency has been achieved. A speed improvement and moreover, the length of the code has been reduced to a mere 88 lines.

In this paper, we present a more general model for topology optimization which resolves this weakness by replacing the equality constraint with a more appropriate inequality constraint. From engineering point of view this is a very important modification because we can replace the inequality constraint with a set of inequality constraints without any difficulty, and therefore the number of interesting points may be arbitrary. The applied model is presented in which displacement-constrained volume-minimizing topology optimization for a two-dimensional (2D) continuum problems is discussed.

In this paper, we present a numerical example using a benchmark model and compare the given results with the those of the original approach. The comparison example is based on the Messerschmitt-Bölkow-Blohm (MBB) beam. The MBB-beam has the function of carrying the floor in the fuselage of an Airbus passenger carrier and became later a basic test example of the topology optimization (see e.g. in Olhoff et al. [8], Zhou and Rozvany [9]).

In the remaining parts of this paper we describe our generalized model in Section 2. In Section 3 a numerical example is presented using the proposed model and the results are compared with the results of the original approach. Finally, some concluding remarks are presented in Section 4.

## 2. MODEL

Without loss of generality, we formulate the proposed new displacement-constrained volume-minimizing topology optimization model with only for two-dimensional (2D) continuum problems. The standard “academic” topology optimization problem of continuum structures can be described as follows:

$$c(\mathbf{x}) = \mathbf{U}'\mathbf{K}\mathbf{U} \rightarrow \min \tag{1}$$

$$V(\mathbf{x}) = \varphi V_0 \tag{2}$$

$$\mathbf{K}\mathbf{U} = \mathbf{F} \tag{3}$$

$$\mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \tag{4}$$

where  $c$  is the compliance,  $\mathbf{U}$  and  $\mathbf{F}$  are the global displacement and load vectors, respectively,  $\mathbf{K}$  is the global stiffness matrix,  $\mathbf{x}$  is the vector of design variables (the element densities),  $V(\mathbf{x})$  and  $V_0$  are the material volume and design domain volume, respectively, and  $\varphi$  is the pre-setting volume fraction. The 2D design domain is assumed to be rectangular and discretized by  $n = e^x \times e^y$  square elements with four nodes per element and two degrees of freedoms (DOFs) per node. Both nodes and elements are numbered column-wise from left to right, and the DOFs  $2i - 1$  and  $2i$  correspond to the horizontal and vertical displacement of node  $i \in \{1, 2, \dots, n\}$ , respectively. The optimization problem (1-4) can be solved by, for example, the well-known and the most widely used optimality criteria

method or any other appropriate nonlinear solver (see, for example, Liu and Tovar [10]).

Let  $u$  define the displacement value at the point of interest and  $\bar{u}$  its pre-setting maximum allowable value. The presented modified displacement-oriented “engineering” topology optimization problem of continuum structures can be formulated as follows:

$$\varphi = \frac{V(\mathbf{x})}{V_0} \rightarrow \min \quad (5)$$

$$u \leq \bar{u} \quad (6)$$

$$\mathbf{K}\mathbf{U} = \mathbf{F} \quad (7)$$

$$\mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \quad (8)$$

where  $\varphi$  is the material volume fraction,  $\mathbf{U}$  and  $\mathbf{F}$  are the global displacement and load vectors, respectively,  $\mathbf{K}$  is the global stiffness matrix,  $\mathbf{x}$  is the vector of design variables (the element densities), and relation  $u \leq \bar{u}$  is the displacement constraint in the point of interest. We have to note, that in the original Yi and Sui [1] model the displacement constraint was formulated as an equality relation, which practically means that the number of “interesting points” is exactly one. The modification resolves this weakness which from engineering point of view a very important result because we can replace the equality constraint with a set of inequality constraints:  $\mathbf{u} \leq \bar{\mathbf{u}}$ . The other very important fact, that the modified displacement-oriented model can be extended very easily to handle stress-oriented relations, which will be demonstrated in the forthcoming paper. Naturally, the more general theoretical model needs more sophisticated numerical method. Therefore, we have to replace the original “optimality-criteria-like” solution searching process with a standard nonlinear programming approach which is able to handle linear (nonlinear) objectives with linear (nonlinear) equality (inequality) constraints. For the numerical treatment of the example presented in Section 3 the Matlab `fmincon` solver was used. According to the original Yi and Sui paper [1], the displacement value  $u = u(\mathbf{x})$  and its gradient  $\partial u / \partial \mathbf{x}$  at the point of interest are defined in the function of the element densities  $\mathbf{x}$ .

### 3. EXAMPLE

The MBB beam is a classical problem in topology optimization. In accordance with the original paper (Sigmund [6]), the MBB beam is used here as an example. The design domain, the boundary conditions, and the external load for the MBB beam are shown in Fig. 1. In this example, our goal is the following: we try to find a volume-fraction-minimal material distribution which satisfies the displacement constraint at the point of interest with tolerance  $\tau = 0.001$ . According to the original Yi and Sui paper [1], the point of interest is taken as the loading point to give comparable results. The example, as shown in Fig. 1, is the half of MBB-beam where a unit load acting down in the left-top corner of the design domain. The ground structure consists of  $n = e^x \times e^y = 60 \times 20 = 1200$  elements, the Young's modulus is  $E_0 = 1$ , the Poisson's ratio is  $\nu = 0.3$ , and the starting volume fraction, used in the original problem solving process, is  $\varphi_0 = 0.50$ . The penalization power is  $p = 3$  and we

use sensitivity filtering with filter radius  $r_{\min} = 1.5$ .

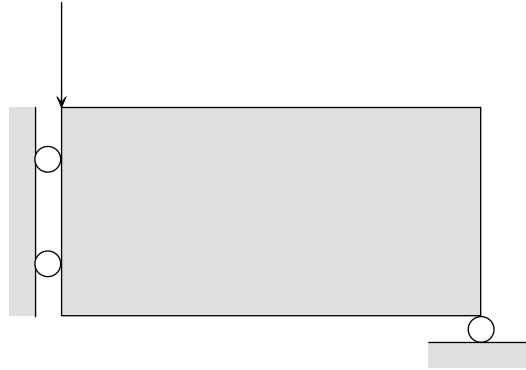


Figure 1. The half design domain, boundary conditions, and external load for the optimization of the symmetric MBB beam

The optimal design of the original and the enhanced approach are shown in Fig. 2-3. When we try to compare the two designs then our first impression is that these designs are practically the same. Fortunately, Table 1 may help to detect the real differences between the two designs. Easy to see, that the results of the more flexible enhanced approach are significantly better than the results given by the original approach. In Table 1, the weight of the structure which is denoted by  $w$  is presented only to ensure a fair comparison between the two approaches because the original optimization problem was formulated as a weight-minimization problem. Naturally, in the presented example there is a trivial relation between the structure weight and volume therefore the result of the weight-minimization and the volume-minimization is the same. According to Table 1, in every case the  $\bar{u} = 203298$  setting was used to ensure the comparability requirement.

The state-of-the-art Matlab **fmincon** solver, which was used to solve the enhanced problem, is able to handle linear (nonlinear) objectives with linear (nonlinear) equality (inequality) constraints where the nonlinear constraint can be defined as a standard Matlab function. In the solution searching process, as a searching tool, the nonlinear interior point solver was selected. It is gratefully acknowledge, that the nonlinear constraint handling function with straightforward extension to compute gradient  $\partial u / \partial \mathbf{x}$  was given from the original **FE** (Finite Element analysis) Matlab function written by Yi and Sui [1].



Figure 2. Result of the original Sui-Yi model



Figure 3. Result of the enhanced model

Table 1: Comparison of the results

<i>Model</i>	$\varphi$	$w$	$u$	$\bar{u}$	$\tau$
<i>Yi &amp; Sui</i>	0.50	600	203.296	203.298	0.002
<i>Csébfalvi</i>	0.45	540	203.297	203.298	0.001

#### 4. CONCLUSION

In this paper, we present a displacement-constrained volume-minimizing topology optimization model for two-dimensional continuum problems. The new model is a generalization of the displacement-constrained volume-minimizing model developed by Yi and Sui [1] in which the displacement is constrained in the loading point. In the original model the displacement constraint was formulated as an equality relation, which practically means that the number of “interesting points” may be exactly one. The recent model resolves this weakness replacing the equality constraint with an inequality constraint. From engineering point of view it is a very important result because we can replace the inequality constraint with a set of inequality constraints without any difficulty. The other very important fact, that the modified displacement-oriented model can be extended very easily to handle stress-oriented relations, which will be demonstrated in the forthcoming paper. Naturally, the more general theoretical model needs more sophisticated numerical problem handling method. Therefore, we replaced the original “optimality-criteria-like” solution searching process with a standard nonlinear programming approach which is able to handle linear (nonlinear) objectives with linear (nonlinear) equality (inequality) constraints. The efficiency of the new approach is demonstrated by an example investigated by several authors. The presented example with reproducible numerical results as a benchmark problem may be used for testing the quality of exact and heuristic solution procedures to be developed in the future for displacement-constrained volume-minimization problems.

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