



## TOPOLOGICAL OPTIMIZATION OF VIBRATING CONTINUUM STRUCTURES FOR OPTIMAL NATURAL EIGENFREQUENCY

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### ABSTRACT

Keeping the eigenfrequencies of a structure away from the external excitation frequencies is one of the main goals in the design of vibrating structures in order to avoid risk of resonance. This paper is devoted to the topological design of freely vibrating continuum structures with the aim of maximizing the fundamental eigenfrequency. Since in the process of topology optimization some areas of domain can potentially be removed, it is quite possible to encounter the problem of localized modes. Hence, the modified Solid Isotropic Material with Penalization (SIMP) model is here used to avoid artificial modes in low density areas. As during the optimization process, the first natural frequency increases, it may become close to the second natural frequency. Due to lack of the usual differentiability of the multiple eigenfrequencies, their sensitivity are calculated by the mathematical perturbation analysis. The optimization problem is formulated by a variable bound formulation and it is solved by the Method of Moving Asymptotes (MMA). Two dimensional plane elasticity problems with different sets of boundary conditions and attachment of a concentrated nonstructural mass are considered. Numerical results show the validity and supremacy of this approach.

**Keywords:** topology optimization, SIMP, multiple eigenvalues, bound formulation, MMA.

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### 1. INTRODUCTION

Topology optimization of continuum structures has a great impact in the field of the structural optimization. Application of topology optimization are shown in [1]. For structural topology optimization design several optimization approach such as the homogenization method [2, 3], the solid isotropic material with penalization (SIMP) [4, 5] and the

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evolutionary structural optimization [6, 7] have been developed. In the homogenization method, microstructures are introduced to the material of the finite elements of the discretized domain and the parameters of these microstructures are treated as the design variables of the optimization problem. Alternatively, the SIMP model uses the material density of each element as the design variable. This can also be related to some geometrical parameters to create some sort of artificial microstructures [8].

Problems of vibration and noise control design are of a great importance in many engineering fields. In classical topology optimization stiffest structure is considered as the objective function. Structures with high first natural frequency tend also to be stiff [9]. Also, maximizing the fundamental or higher order frequency, can be offered to avoid resonance in problems [10]. However, the number of paper that deal with topology optimization of dynamic problem is limited. The first attempt at eigenvalue topology optimization dealt with the reinforcement of given 2D structures was considered by Diaz and Kikuchi [11]. Tenek and Hagiwara [12] dealt with maximizing the eigenfrequencies of plates using the homogenization method and mathematical programming. The problem objective function is defined as scalar weighted sum of the first five eigenfrequencies; see [13-16].

Two main subjects arise from eigenvalue optimization. Artificial modes related to low density areas are one of the main problems in optimization frequencies or buckling loads [17, 18]. Several paper include a technique to remove artificial localized modes [7, 10, 18]. Second is the possibility of multiple eigenvalue problem. Such eigenvalues are not usual differentiable. Sensitivities calculation of repeated eigenvalues are available in [19, 20].

Ma et al. [14], and Jog [21] dealt with topology optimization for minimum dynamic compliance of continuum structure subjected to force vibration. In the paper of Alavi et al. [22], topological design of structures under transient loads are presented. Maximization of the gap between two adjacent frequencies have been considered in the papers [23, 24]. Topology optimization has also been applied to maximize natural frequency of two-dimensional structures with an additional non-structural concentrated mass [25, 26].

The bound formulation which eases the proper treatment of multiple eigenvalues has been formulated in references [10, 23, 27, 28].

In this paper, we present topology optimization of fundamental eigenfrequency of two dimensional structures with a concentrated mass. Also possibility of multiple eigenfrequencies is taken into consideration and Sensitivity of them are computed by the results of the mathematical perturbation analysis approach [19]. Spurious modes related to subregions with low values of material density are captured by using the modified SIMP [10]. The topology optimization is formulated by a bound formulation and the problem of eigenvalue topology optimization is solved by the Method of Moving Asymptotes (MMA) [29].

## 2. MATERIAL INTERPOLATION SCHEME

### 2.1 The SIMP model

The purpose of the topology optimization process is to find the void-solid distribution of a given amount of material. By considering isotropy for the solid part of the structure, the element elasticity matrix  $\mathbf{E}_e$  may be considered as

$$\mathbf{E}_e(x_e) = x_e \mathbf{E}_e^0, \quad (0 < x_{\min} \leq x_e \leq 1), \quad (1)$$

where  $x_e$  and  $\mathbf{E}_e^0$  are, respectively, the element material density and elasticity matrix of homogeneous solid.  $x_{\min}$  is a lower bound for  $x_e$ . To prevent singularity  $x_{\min}$  is not zero. Although (1) yields a relaxed optimization algorithm, it results in some porous areas in the optimum structure [2]. It is more practical to achieve a solution which be composed of solid and void region. To attain this aim, it may be desirable to penalizing the intermediate values for  $x_e$ . SIMP model [4, 5] can easily be provided by changing (1) to the form

$$\mathbf{E}_e(x_e) = x_e^p \mathbf{E}_e^0, \quad (2)$$

where  $p$  is the penalization factor and is usually 3. Also the element mass matrix may be considered as

$$\mathbf{M}_e(x_e) = x_e^q \mathbf{M}_e^0, \quad (3)$$

where  $\mathbf{M}_e^0$  is the mass matrix corresponding to the element with fully solid material, and usually the  $q = 1$ .

The global stiffness matrix and mass matrix can be expressed as

$$\mathbf{K} = \sum_1^n x_e^p \mathbf{K}_e^0, \quad (4)$$

$$\mathbf{M} = \sum_1^n x_e^q \mathbf{M}_e^0, \quad (5)$$

where  $n$  and  $\mathbf{K}_e^0$  are respectively the number of elements and stiffness matrix corresponding to the element with fully solid material.

## 2.2 The approach of removing localized modes

In the present paper the eigenvalue optimization problem is not formulated as a reinforcement of an existing structure, so there is a problem associated with spurious localized modes in low density regions [18]. If  $p = 3$  and  $q = 1$  the SIMP model for eigenvalue topology optimization may cause artificial eigenmodes related to very small corresponding eigenfrequencies. Following Du and Olhof [10] we may replace (3) by the modified SIMP model

$$\mathbf{M}_e(x_e) = \begin{cases} x_e \mathbf{M}_e^0, & x_e \geq 0.1 \\ (c_1 x_e^6 + c_2 x_e^7) \mathbf{M}_e^0, & x_e \leq 0.1 \end{cases}, \quad (6)$$

where the two coefficients  $c_1 = 6 \times 10^5$  and  $c_2 = -5 \times 10^6$  enforce the  $C^0$  and  $C^1$  continuity at the value  $x_e = 0.1$  of the element material density. By using the modified SIMP model as stated above, the outcome will be that the artificial localized modes are eliminated. The considered approach has applied in numerical examples in section 4.

### 3. FORMULATION OF EIGENVALUE TOPOLOGY OPTIMIZATION

#### 3.1 General problem of optimization

Supposing the damping term is dispensable, the dynamic behavior of continuum structure in the finite element can be written as

$$(\mathbf{K} - \omega_i^2 \mathbf{M})\mathbf{u}_i = 0, \quad (7)$$

where  $\omega_i$  is  $i$ th natural frequency and  $u_i$  is the corresponding eigenvector.

The eigenvalue optimization problem can be considered as max-min formulation

$$\text{find } \mathbf{x} = \{x_1, x_2, \dots, x_n\}, \quad (8a)$$

$$\text{max } \left\{ \min_{i=1, \dots, N} \{\omega_i^2\} \right\}, \quad (8b)$$

$$\text{S.t. } \mathbf{K}\mathbf{u}_i = \omega_i^2 \mathbf{M}\mathbf{u}_i, \quad i = 1, \dots, N, \quad (8c)$$

$$\mathbf{u}_i^T \mathbf{M}\mathbf{u}_j = \delta_{ij}, \quad i \geq j, \quad i, j = 1, \dots, N, \quad (8d)$$

$$\sum_{e=1}^n x_e - fV_0 \leq 0, \quad (8e)$$

$$\mathbf{0} < \mathbf{x}_{\min} \leq \mathbf{x} \leq \mathbf{1}, \quad (8f)$$

In these equations,  $N$  is total number of degrees of freedom of the admissible design domain,  $f$  and  $V_0$  are respectively the prescribed volume fraction and design domain volume,  $\mathbf{X}$  denotes the vector of element material densities,  $x_{\min}$  represent a vector of lower bound for  $\mathbf{X}$  (a non-zero vector to prevent singularity).

#### 3.2 Variable bound optimization

In this paper we reformulate the min-max problem using bound formulation that has an advantage when we have multiple eigenfrequencies. This approach leads to much more considerable results than the usual scalar weighted sum of multiobjective problems [28]. In this formulation a scalar variable  $Z$  is taken simultaneously to be objective function and a variable lower limit for first and higher order eigenvalues (depend on possible multiplicity). Following Du and Olhof [10] and Svanberg [30] the optimization problem can be expressed as

$$\text{find } \mathbf{x} = \{x_1, x_2, \dots, x_n\}, \quad (9a)$$

$$\text{max } Z, \quad (9b)$$

$$\text{S.t. } Z - \omega_i^2 \leq 0, \quad i = 1, 2, \dots, J+1, \quad (9c)$$

$$\mathbf{K}\mathbf{u}_i = \omega_i^2 \mathbf{M}\mathbf{u}_i, \quad i = 1, \dots, J+1, \quad (9d)$$

$$\mathbf{u}_i^T \mathbf{M}\mathbf{u}_j = \delta_{ij}, \quad i \geq j, \quad i, j = 1, \dots, J+1, \quad (9e)$$

$$\text{Constraints: } 8(e) - 8(f), \quad (9f)$$

In this equation we considered  $J$  to be the number of repeated frequencies  $\omega_1 = \dots = \omega_J$ . By using the scalar variable  $Z$  even if repeated eigenvalues are available, the optimization problem is differentiable if they are considered as problems in all variables [10, 28].

### 3.3 Sensitivity calculation of simple eigenvalue

We here determine the sensitivity of the unimodal eigenvalue  $\lambda_i = \omega_i^2$ . If we assume the eigenfrequency is simple, then the corresponding eigenvector will be unique. Therefore it is differentiable with respect to design variables. To calculate the sensitivity of unimodal eigenvalue we differentiate (8c) with respect to  $x_e$ , and achieve

$$\frac{\partial \mathbf{K}}{\partial x_e} \mathbf{u}_i + (\mathbf{K} - \lambda_i \mathbf{M}) \frac{\partial \mathbf{u}_i}{\partial x_e} = \frac{\partial \lambda_i}{\partial x_e} \mathbf{M}\mathbf{u}_i + \lambda_i \frac{\partial \mathbf{M}}{\partial x_e} \mathbf{u}_i, \quad e = 1, \dots, n, \quad (10)$$

Premultiplying by  $\mathbf{u}_i^T$  and applying the equation (7) and the normalization equation (8d), yields [19]

$$\frac{\partial \lambda_i}{\partial x_e} = \mathbf{u}_i^T \left( \frac{\partial \mathbf{K}}{\partial x_e} - \lambda_i \frac{\partial \mathbf{M}}{\partial x_e} \right) \mathbf{u}_i, \quad e = 1, \dots, n, \quad (11)$$

The derivative of  $\mathbf{K}$  and  $\mathbf{M}$  matrices can be computed from equations (4) and (6), i.e.

$$\frac{\partial \mathbf{K}}{\partial x_e} = \frac{\partial (\sum_{e=1}^n x_e^p \mathbf{K}_e^0)}{\partial x_e} = p x_e^{(p-1)} \mathbf{K}_e^0, \quad e = 1, \dots, n, \quad (12)$$

$$\frac{\partial \mathbf{M}}{\partial x_e} = \begin{cases} \mathbf{M}_e^0, & x_e > 0.1 \\ (6c_1 x_e^5 + 7c_2 x_e^6) \mathbf{M}_e^0, & x_e \leq 0.1 \end{cases}, \quad e = 1, \dots, n, \quad (13)$$

Consequently, the sensitivity of  $i$ 'th natural frequency with respect to particular design variable  $x_e$  becomes

$$\frac{\partial \lambda_i}{\partial x_e} = \begin{cases} \mathbf{u}_i^T (px_e^{(p-1)} \mathbf{K}_e^0 - \lambda_i \mathbf{M}_e^0) \mathbf{u}_i, & x_e > 0.1, \\ \mathbf{u}_i^T (px_e^{(p-1)} \mathbf{K}_e^0 - \lambda_i (6c_1 x_e^5 + 7c_2 x_e^6) \mathbf{M}_e^0) \mathbf{u}_i, & x_e \leq 0.1, \end{cases} \quad e = 1, \dots, n, \quad (14)$$

### 3.4 Sensitivity calculation of multiple eigenvalues

During the maximization, the first eigenfrequency may become close with its adjacent eigenfrequency, which is mentioned to multiple eigenvalues. The eigenvectors of the multiple eigenvalues are not unique. Any linear combination of eigenvectors is also an eigenvector and will satisfy Equation (7). The new sensitivity formulation is based on the result of the mathematical perturbation analysis of the repeated eigenfrequency and the corresponding eigenvectors [19]. Following Seyranian et al. [19] the sensitivities of the multiple eigenvalue  $\bar{\lambda}$  with respect to changes of a single design parameter  $x_e$  can be considered as the eigenvalues of a  $J$ -dimensional subeigenvalue problem as

$$\det[\mathbf{u}_s^T (\frac{\partial \mathbf{K}}{\partial x_e} - \bar{\lambda} \frac{\partial \mathbf{M}}{\partial x_e}) \mathbf{u}_k - \mu \delta_{sk}] = 0, \quad s, k = 1, \dots, J, \quad e = 1, \dots, n, \quad (15)$$

where as in the unimodal case, the derivative of  $\mathbf{K}$  and  $\mathbf{M}$  matrices can be expressed by Equations (12) and (13), respectively.

### 3.5 Method of solution

In order to solve the eigenvalue topology optimization several approaches such as optimality criteria (OC) method [14], the method of moving asymptotes (MMA) [29], mathematical programming (MP) [31] and less mathematically rigorous approaches such as evolutionary method [32] can be used. In the present paper to solve variable bound optimization of eigenvalue problem we use MMA [29] which has been proven to be amongst the most effective methods [33]. Also, MMA is a mathematical gradient base approach, it is well matched with the large number of topology and shape design variables [34].

## 4. SUMMARY OF OPTIMIZATION PROCEDURE

The proposed iterative procedure is illustrated in a flowchart of Fig. 1. The bound variable formulation of eigenvalue topology optimization has been coded in the MATLAB software. Checkerboard and mesh-dependency problems usually arise in topology optimization problem. To prevent these problems, we have used the mesh-independent filter as described in [5] by weighted averaging of sensitivities over the neighbouring elements (see e.g. Hassani and Hinton [35]). The algorithm repeats until the maximum of absolute relative change in design variables or objective function in two adjacent iteration is less than a prescribed value.

### 5. NUMERICAL EXAMPLES

In this section two illustrative examples is presented which shows efficiency of the proposed method. All of the examples are modeled by four-node plane stress elements.

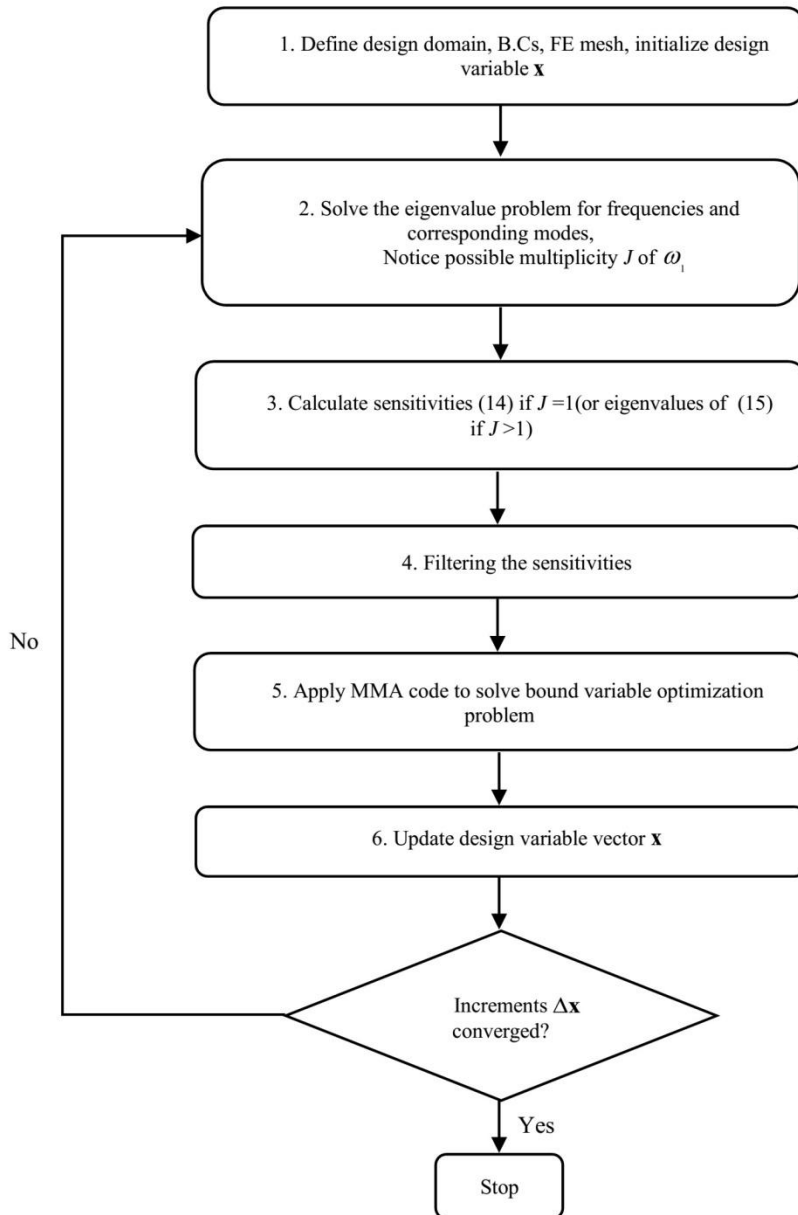


Figure 1. Flowchart of the proposed optimization algorithm

#### 5.1 A rectangle clamped beam

As a first example, a rectangle beam with clamped ends, as shown in Fig. 2, is considered. A non-structural lumped mass posed on the four elements in the center of design domain. The

geometrical and physical properties of the system are given in Table 1. The first natural frequency of design domain with fully solid material is  $\omega_1 = 154$  rad/s. The objective is to find topological design of structure for maximum fundamental frequency with 50.29% material of the design domain. The optimal design is shown in Fig. 3a. Fig. 3b shows the result in Zhao et.al [26]. The first frequency of optimal design in the present study is 153 rad/s. which is higher than the result in Zhao et.al [26], which shows the superiority of approach used in this paper. Also the topological design in Fig. 3a is more practical than the optimal design in Fig. 3b.

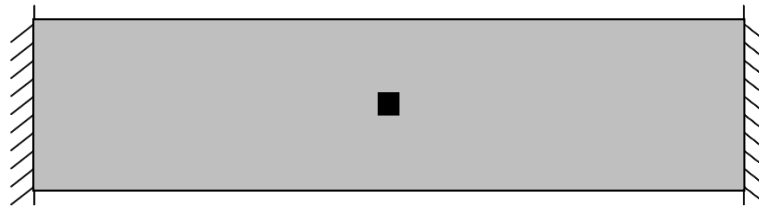


Figure 2. A clamped beam

Table 1: Physical and geometrical properties of the clamped beam

Parameter	Value	Unit
Young's modulus	25	GPa
poisson's ratio	0.3	---
mass density	2500	kg
lumped mass	125	kg
Beam length	14	m
Beam width	2	m
Beam thickness	0.01	m



(a)



(b)

Figure 3. Topological design of the clamped beam with an additional non-structural mass. a. present study b. Evolutionary approach [25]



5.2 A rectangle cantilever beam

As shown in Fig. 4, a cantilever beam with material and geometrical properties given in table 2 is presented. A concentrated non-structural mass posed on the middle of right side. The volume fraction is 40%. The existing material is uniformly distributed over the design domain. The resulted topological design by the proposed method is shown in Fig 5. The fundamental frequency of the above optimal topology is 12.92 Hz. The result is more optimized than one obtained by Zheng et.al [25].



Figure 4. A rectangle cantilever beam

Table 2: Material and geometrical properties of the cantilever beam

Parameter	Value	Unit
Young's modulus	10	GPa
poisson's ratio	0.3	---
mass density	1000	kg
lumped mass	16000	kg
Beam length	8	m
Beam width	4	m
Beam thickness	1	m



Figure 5. Optimal design of the cantilever beam with a concentrated mass attached on the middle of right side

## 6. CONCLUSION

In the present research the MMA is employed to solve bound variable formulation of eigenvalue topology optimization. The modified SIMP model has been used to handle the localized modes problem associated with low density areas. The optimization problem is reformulated by the bound variable formulation in order to facilitate the treatment of bimodal eigenfrequencies. Some illustrative examples of the eigenvalue topological design for two-dimensional plane elasticity problems with an additional concentrated non-structural mass is presented. The results show the superiority of the procedure used in this paper. The results also demonstrate that by employing the proposed approach checkerboard patterns can be avoided and the resulted topological designs are more practical than those obtained by using other optimization techniques.

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