A NEW APPROACH FOR SOLVING FULLY FUZZY QUADRATIC PROGRAMMING PROBLEMS

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ABSTRACT

Quadratic programming (QP) is an optimization problem wherein one minimizes (or maximizes) a quadratic function of a finite number of decision variable subject to a finite number of linear inequality and/or equality constraints. In this paper, a quadratic programming problem (FFQP) is considered in which all cost coefficients, constraints coefficients, and right hand side are characterized by L-R fuzzy numbers. The FFQP problem is converted into the fully fuzzy linear programming using the Taylor series and hence into a linear programming problem which may be solved by applying GAMS Software. Finally, an example is given to illustrate the practically and the efficiency of the method.

Keywords: Fully Fuzzy Quadratic; L – R Fuzzy Numbers; Taylor Series; Fully Fuzzy Linear Programming; Ranking Function; Linear Programming; Fuzzy Optimal Solution.

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1. INTRODUCTION

Quadratic programming (QP) is an optimization problem whose objective function is quadratic function and the constraints are linear equalities or inequalities. QP is widely used in real world problem to optimize portfolio selection problem, in the regression to perform the least square method, in chemical plants to control scheduling, in the sequential quadratic programming, economics, engineering design etc. Since the QP is the most interesting class of the optimization, so it is known as the NP-hard. There are several method and algorithms for solving the QP problem introduced by Pardalos and Rosen [1], Horst and Tuy [2] and Bazaraa et al. [3]. Beck and Teboulle [4] established a necessary global optimality condition for the nonconvex QP optimization problem with binary constraints. Kochenberger et al. [5] studied the unconstrained binary quadratic programming problem. Xia [6] explored local optimality conditions for obtaining new sufficient optimality conditions for nonconvex quadratic optimization with binary constraints. Bonami et al. [7] introduced an effective


In this paper, FFQP problem is studied. Proposed approach is applied for obtaining the optimal compromise solution for the problem without converting it into the corresponding deterministic problem.

The outlay of the paper is organized as follows: Section2 presented some preliminaries related to the L – R fuzzy numbers and their arithmetic operations. Section3 formulates fully fuzzy quadratic programming problem. In Section4, a proposed approach is applied for solving the problem introduced in Section2. In Section 5, a numerical example to illustrate the efficiency of the solution approach is given. Finally, some concluding remarks are reported in Section 6.
2. PRELIMINARIES

This section introduces some of basic concepts and results related to \( L - R \) fuzzy numbers and their arithmetic operations.

**Definition 1**[27]. A triangular fuzzy number can be represented completely by a triplet \( \tilde{A} = (a, b, c) \), and has membership

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & x < a, \\
\frac{x - a}{b - a}, & a \leq x \leq b, \\
\frac{c - x}{c - b}, & b \leq x \leq c, \\
0, & x > c.
\end{cases}
\]  

(1)

**Definition 2**. The ordinary representative of the fuzzy number \( A = (a, b, c) \) is given by

\[ \hat{A} = \frac{a + 2b + c}{4}. \]

Figure 1. \( L-R \) Fuzzy number representation.

**Definition 3**[28]. A fuzzy number \( \tilde{A} = (x, \alpha, \beta)_{LR} \) is said to be an \( L - R \) fuzzy number if

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\mathcal{L}\left(\frac{m - x}{\alpha}\right), & x \leq m, \alpha > 0 \\
\mathcal{R}\left(\frac{x - m}{\beta}\right), & x \geq m, \beta > 0
\end{cases}
\]  

(2)

where \( m \) is the mean value of \( \tilde{A} \) and \( \alpha \) and \( \beta \) are left and right spreads, respectively, and a function \( \mathcal{L}(\cdot) \) is a left shape function satisfying: \( \mathcal{L}(x) = \mathcal{L}(-x), \mathcal{L}(0) = 1, \mathcal{L}(x) \) is nonincreasing on \([0, \infty[). \)

It is noted that a right shape function \( \mathcal{R}(\cdot) \) is similarly defined as \( \mathcal{L}(\cdot) \).

For two \( L - R \) fuzzy numbers \( \tilde{A} = (x, \alpha, \beta)_{LR} \) and \( \tilde{B} = (y, \gamma, \sigma)_{LR} \), the arithmetic operations are:
1. Addition: $\tilde{A} \oplus \tilde{B}$

$$(x, \alpha, \beta)_{LR} \oplus (y, \gamma, \sigma)_{LR} = (x + y, \alpha + \gamma, \beta + \sigma)_{LR} \quad (3)$$

2. Opposite: $-\tilde{A}$

$$-(x, \alpha, \beta)_{LR} = (-x, \alpha)_{LR} \quad (4)$$

3. Subtraction: $\tilde{A} \ominus \tilde{B}$

$$(x, \alpha, \beta)_{LR} \ominus (y, \gamma, \sigma)_{RL} = (x - y, \alpha + \sigma, \beta + \gamma)_{LR} \quad (5)$$

4. Multiplication: $\tilde{A} \bigodot \tilde{B}$

$$\tilde{A} \bigodot \tilde{B} = \begin{cases} (xy, x\gamma + y\alpha, x\sigma + y\beta)_{LR}, & \text{if } \tilde{A} > 0, \tilde{B} > 0; \\ (xy, y\alpha - x\sigma, y\beta - x\sigma)_{RL}, & \text{if } \tilde{A} < 0, \tilde{B} > 0; \\ (xy, -y\beta - x\sigma, -y\sigma - x\gamma)_{RL}, & \text{if } \tilde{A} < 0, \tilde{B} < 0. \end{cases} \quad (6)$$

5. Scalar multiplication: $\lambda \bigotimes \tilde{A}$

$$\lambda \bigotimes (x, \alpha, \beta)_{LR} = \begin{cases} (\lambda x, \lambda \alpha, \lambda \beta)_{LR}, & \lambda > 0; \\ (\lambda x, -\lambda \beta, -\lambda \alpha)_{RL}, & \lambda < 0. \end{cases} \quad (7)$$

3. PROBLEM FORMULATION AND SOLUTION CONCEPTS

A fully fuzzy quadratic programming problem is formulated in matrix form as follows

$$\min \tilde{Z} = \tilde{c}^T \bigotimes X \bigoplus \frac{1}{2} X^T \tilde{Q} \bigotimes X \quad (8)$$

Subject to:

$$\tilde{A} \bigotimes \bigotimes X \bigotimes (\preceq) b, X \succeq 0. \quad (9)$$

Problem (8)-(9), can be rewritten in the following compact form as

$$\min Z = \sum_{j=1}^n \tilde{c}_j \bigotimes x_j \bigoplus \frac{1}{2} \left( \sum_{i=1}^n \sum_{j=1}^n x_i \tilde{q}_{ij} \bigotimes x_j \right) \quad (10)$$

Subject to:
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\[ x \in \mathcal{M} = \left\{ \sum_{j=1}^{n} a_{ij} \bigoplus x_j \preceq b_i, i = 1, 2, \ldots, m, \sum_{j=1}^{n} a_{ij} \bigotimes x_j \succeq b_i, i = m_1, \ldots, m \right\} \quad (11) \]

where, \( \tilde{C} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n) \), and \( \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) \) are fuzzy cost vector and fuzzy right-hand side vector. \( X = (x_1, x_2, \ldots, x_n) \) is a vector of variables, and also \( \tilde{Q} = [q_{ij}]_{n \times n} \) is a matrix of quadratic form which is symmetric and positive semi definite, and \( \tilde{A} = [a_{ij}]_{m \times n} \). It is assumed that all of \( \tilde{A}, \tilde{b}, \tilde{C}, \) and \( \tilde{Q} \in F(R) \), where \( F(R) \) denotes the set of all \( L - R \) fuzzy numbers.

**Definition 4.** A non-negative fuzzy vector \( x_j \) is said to be fuzzy feasible solution for problem \((10)\)- \((11)\) if it satisfies the constraints \((11)\).

**Definition 5.** A fuzzy feasible solution \( x_j^* \) is called a fuzzy optimal solution for problem \((10)\)- \((11)\), if

\[
\begin{aligned}
\sum_{j=1}^{n} \tilde{c}_j \bigoplus x_j^* \preceq \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i^* \bigotimes q_{ij} \bigotimes x_j^*\right) \\
\sum_{j=1}^{n} \tilde{c}_j \bigoplus x_j \bigoplus \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i \bigotimes q_{ij} \bigotimes x_j\right); \forall x_j
\end{aligned}
\]

\[ (12) \]

Let \( \tilde{c}_j, \tilde{q}_{ij}, \tilde{a}_{ij} \), and \( \tilde{b}_i \) represent by the \( L - R \) fuzzy numbers, \( (c_j, \alpha_j, \beta_j)_{LR} \), \( (q_{ij}, \delta_{ij}, \epsilon_{ij})_{LR} \), \( (a_{ij}, \epsilon_{ij}, \theta_{ij})_{LR} \), and \( (b_i, \theta_i, \rho_i) \), respectively. Then problem \((10)\) - \((11)\) becomes

\[
\begin{aligned}
\min \, \tilde{Z} = \sum_{j=1}^{n} (c_j, \alpha_j, \beta_j)_{LR} \bigotimes x_j \bigotimes \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i \bigotimes (q_{ij}, \delta_{ij}, \epsilon_{ij})_{LR} \bigotimes x_j\right)
\end{aligned}
\]

\[ (13) \]

Subject to:

\[
\begin{aligned}
x \in \tilde{M} = \left\{ \sum_{i=1}^{n} (a_{ij}, \epsilon_{ij}, \theta_{ij})_{LR} \bigotimes x_i \preceq (b_i, \theta_i, \rho_i)_{LR}, i = 1, 2, \ldots, m, \right. \\
\left. \sum_{i=1}^{n} (\tilde{a}_{ij}, \tilde{\epsilon}_{ij}, \tilde{\theta}_{ij})_{LR} \bigotimes x_i \succeq (\tilde{b}_i, \tilde{\theta}_i, \tilde{\rho}_i)_{LR}, i = m_1, \ldots, m \right\}; \forall x_j \geq, j = 1, 2, \ldots, n
\end{aligned}
\]

\[ (14) \]

**Definition 6** [29]. Assume that function \( G \) has first order partial derivatives (i. e., is of
class $C^{(1)}$. The first two term of the Taylor series generated by $G(x_1, x_2, ..., x_n)$ at $B(a_1, a_2, ..., a_n)$ is

$$G(B) + \frac{\partial}{\partial x_1} G(B)(x_1 - a_1) + \frac{\partial}{\partial x_2} G(B)(x_2 - a_2) + ... + \frac{\partial}{\partial x_n} G(B)(x_n - a_n) = 0 \quad (15)$$

### 4. PROPOSED APPROACH

In this section, the steps of the proposed approach for solving fully fuzzy quadratic programming is illustrated as

**Step 1:** Consider the fully fuzzy quadratic problem (12)-(13) and choose any arbitrary feasible non-zero point initially, say $\bar{x}$.

**Step 2:** Using the definition, expand the objective function (12) to the Taylor series at the $\bar{x}$, so as to obtain fully fuzzy linear programming (FFLP) problem.

**Step 3:** Consider the FFLP problem

$$\min \bar{W} = \sum_{j=1}^{n} (d_j^l, q_j, \omega_j)_{LR} \bigodot x_j \quad (16)$$

Subject to:

$$x \in \bar{M} = \left\{ \begin{array}{l} \sum_{j=1}^{n} (a_{ij}, \epsilon_{ij}, \theta_{ij})_{LR} \bigodot x_j \leq (b_i, \varphi_i, \rho_i)_{LR} \quad , i = 1, 2, ..., m, \\ \sum_{j=1}^{n} (a_{ij}, \epsilon_{ij}, \theta_{ij})_{LR} \bigodot x_j = (b_i, \varphi_i, \rho_i)_{LR} \quad , i = m_{i+1}, ..., m \\ x_j \geq , j = 1, 2, ..., n \end{array} \right\} \quad (17)$$

Rewrite the problem (16) as in the following form

$$\min \tilde{W} = \sum_{j=1}^{n} (d_j^l, d_j^c, d_j^u) \bigodot x_j \quad (18)$$

Subject to:
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\[ x \in \tilde{M} = \left\{ \begin{array}{l}
\sum_{j=1}^{n} (a_{ij}^l, a_{ij}^c, a_{ij}^u) \odot x_j \leq (b_i^l, b_i^c, b_i^u), i = 1, 2, ..., m_1, \\
\sum_{j=1}^{n} (a_{ij}^l, a_{ij}^c, a_{ij}^u) \odot x_j \approx (b_i^l, b_i^c, b_i^u), i = m_i + 1, ..., m \\
\sum_{j=1}^{n} (a_{ij}^l, a_{ij}^c, a_{ij}^u) \odot x_j \geq 1, j = 1, 2, ..., n
\end{array} \right\} \] (19)

Based on the ranking function in Definition 2, convert the problem (18) into the following linear programming problem as:

\[ \min \tilde{W} = \sum_{j=1}^{n} d_j^l + 2d_j^c + d_j^u x_j \] (20)

Subject to

\[ x \in \tilde{M} = \left\{ \begin{array}{l}
\sum_{j=1}^{n} \frac{a_{ij}^l + 2a_{ij}^c + a_{ij}^u}{4} x_j \leq \frac{b_i^l + 2b_i^c + b_i^u}{4}, i = 1, 2, ..., m_1, \\
\sum_{j=1}^{n} \frac{a_{ij}^l + 2a_{ij}^c + a_{ij}^u}{4} x_j = \frac{b_i^l + 2b_i^c + b_i^u}{4}, i = m_i + 1, ..., m \\
x_j \geq j = 1, 2, ..., n
\end{array} \right\} \] (21)

Let \( \bar{x} \) be the solution of problem (21).

**Step 4:** Expand the objective function (12) to the Taylor series at the solution \( \bar{x} \).

**Step 5:** Solve the problem resulted from Step 4 subject to the given constraints to obtain another solution \( \bar{x} \).

**Step 6:** If the two solutions \( \bar{x} \) and \( \bar{x} \) overlap then the solution of the problem (20) is obtained, and then calculate the fuzzy objective value and stop. Otherwise, assign \( \bar{x} \) to \( \bar{x} \) and return to the Step 4.

The flow chart of the proposed approach can be illustrated as in the Fig. 2 below.
5. NUMERICAL EXAMPLE

Consider the FFQP problem as

$$\min \tilde{Z} = \left[ (-2, -1, 1)_{LR} \odot x_1 \oplus (-6, 3, 3)_{LR} \odot x_2 \oplus \\
(1, 0, 1)_{LR} \odot (-2, 2, 6)_{LR} \odot x_1 \oplus \\
(-2, -2, 6)_{LR} \odot (3, 7, 3)_{LR} \odot x_1 \right]$$

(22)
Subject to

\[(1,1,1)_{LR} \odot x_1 \oplus (1,2,2)_{LR} \odot x_2 \leq (3,5,1)_{LR},\]
\[(-2,2,6)_{LR} \odot x_1 \oplus (2,1,1)_{LR} \odot x_2 \leq (3,5,1)_{LR},\]
\[(2,2,2)_{LR} \odot x_1 \oplus (1,0,0)_{LR} \odot x_2 \leq (3,2,2)_{LR}, x_1, x_2 \geq 0.\]  

(23)

**Step 1-2:** Choose \((0.5,1)\) as an arbitrary initial feasible non-zero point, and expanding the function to the Taylor series at it as:

\[\frac{\partial \bar{Z}}{\partial x_1} = (-2, -1, 1) \oplus \frac{1}{2} [(2, 0, 2) \odot x_1 \oplus (-4, 4, 12) \odot x_2],\]
\[\frac{\partial \bar{Z}}{\partial x_2} = (-6, 3, 3) \oplus \frac{1}{2} [(-4, 12) \odot x_1 \oplus (6, 14, 6) \odot x_2].\]  

(24)

**Step 3:** Construct the problem

\[\min \bar{Z} = (-3,5,1,5)_{LR} \odot x_1 \oplus (-4,9,9)_{LR} \odot x_2\]  

(25)

Subject to:

\[(1,1,1)_{LR} \odot x_1 \oplus (1,2,2)_{LR} \odot x_2 \leq (3,5,1)_{LR},\]
\[(-2,2,6)_{LR} \odot x_1 \oplus (2,1,1)_{LR} \odot x_2 \leq (3,5,1)_{LR},\]
\[(2,2,2)_{LR} \odot x_1 \oplus (1,0,0)_{LR} \odot x_2 \leq (3,2,2)_{LR}, x_1, x_2 \geq 0.\]  

(26)

Then, problem (25) can be rewritten as:

\[\min \bar{Z} = (-4.5, -3.5, 1.5) \odot x_1 \oplus (-13, -4, 5) \odot x_2\]  

(27)

Subject to:

\[(0,1,2) \odot x_1 \oplus (-1,1,3) \odot x_2 \leq (-2,3,4),\]
\[(-4,-2,4) \odot x_1 \oplus (1,2,3) \odot x_2 \leq (-2,3,4).\]  

(28)
\[(0, 2, 4) \odot x_1 \oplus (1,1,1) \odot x_2 \preceq (1, 3, 5), x_1, x_2 \geq 0.\]

Using the ranking method defined in definition 3, the problem (27) becomes:

\[
\min \hat{Z} = -2x_1 - 3.9975x_2 \tag{29}
\]

Subject to:

\[
x_1 + x_2 \leq 2; \tag{30}
\]

\[
-x_1 + 2x_2 \leq 2;
\]

\[
2x_1 + x_2 \leq 3, x_1, x_2 \geq 0.
\]

The optimal solution of problem (21) is \(\bar{x} = \left(\frac{2}{3}, \frac{4}{3}\right)\).

**Step 4:** Expand the objective function of problem (17) to the Taylor series at the solution \(\bar{x} = \left(\frac{2}{3}, \frac{4}{3}\right)\), to obtain the fully fuzzy linear programming and then linear programming as:

\[
\min \hat{Z} = -2.5x_1 - 2.75x_2 \tag{31}
\]

Subject to:

\[
x_1 + x_2 \leq 2; \tag{32}
\]

\[
-x_1 + 2x_2 \leq 2;
\]

\[
2x_1 + x_2 \leq 3, x_1, x_2 \geq 0.
\]

The optimal solution of problem (21) is \(\bar{x} = \left(\frac{2}{3}, \frac{4}{3}\right)\).

**Step 5:** The two solutions \(\bar{x}\), and \(\bar{x}\) are consistent, the optimal solution is \(x^* = \left(\frac{2}{3}, \frac{4}{3}\right)\), and the fuzzy optimum value is \(Z = (-6.444, 9.556, 12.889)_{LR}\).

5. CONCLUSION

In this paper, a new approach is proposed on solving fully fuzzy quadratic programming in which all the cost coefficients, constraints coefficients, and right hand side are characterized by \(L-R\) fuzzy numbers. The approach can easily be applied to solve any QP problem. Through this approach, the FFQP problem is converted into the fully fuzzy linear
programming and hence into linear programming with an arbitrary initial point. The advantages of the approach are differs from the others methods in computational step, easier than the other method that can be solved algebraically, the final solution can be obtained rapidly, and implemented in various types of nonlinear programming.

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