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# PERFORMANCE OF DIFFERENT ANT-BASED ALGORITHMS FOR OPTIMIZATION OF MIXED VARIABLE DOMAIN IN CIVIL ENGINEERING DESIGNS

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## ABSTRACT

Ant colony optimization algorithms (*ACOs*) have been basically introduced to discrete variable problems and applied to different research domains in several engineering fields. Meanwhile, abundant studies have been already involved to adapt different ant models to continuous search spaces. Assessments indicate competitive performance of *ACOs* on discrete or continuous domains. Therefore, as potent optimization algorithms, it is encouraging to involve ant models to mixed-variable domains which simultaneously tackle discrete and continuous variables. This paper introduces four ant-based methods to solve mixed-variable problems. Each method is based upon superlative ant algorithms in discrete and/or continuous domains. Proposed methods' performances are then tested on a set of three mathematical functions and also a water main design problem in engineering field, which are elaborately subject to linear and non-linear constraints. All proposed methods perform rather satisfactorily on considered problems and it is suggested to further extend the application of methods to other engineering studies.

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## 1. INTRODUCTION

Inter-basins water transfer mains demand significant construction and operational costs. Optimum design of such infrastructures has received considerable attentions during the last decade. Most of these researches have employed relevant techniques to minimize the capital and operational costs associated with large capacity water conveyance systems ([1-4]). Martin [5] successfully used dynamic programming to optimally design a pressurized water main. The selected design specifies the number and the size of pumping stations, diameter, and pressure class of the pipeline at the beginning of each stage interval over the planning period. Afshar et al. [6] developed a *DP* model to optimally integrate hydropower plant into a water supply main. In order to provide a rational basis for narrowing existing potential alternatives into a final alignment corridor, a Geographical Information System (*GIS*) based route selection process was introduced by Luettinger and Clark [7].

During the last decade the evolutionary and meta-heuristic algorithms, such as Genetic Algorithms (*GAs*), Simulated Annealing (*SA*) and more recently Ant Colony Optimization algorithms (*ACOs*) have received considerable attentions. Ant colony optimization algorithms have been applied successfully to solve various combinatorial optimization problems ([8-13]). An overview of ant algorithms may be found in an interesting paper by Dorigo and Di Caro [14]. In the area of water resources and hydraulics engineering, Abbaspour et al. [15] employed *ACO* algorithms to estimate hydraulic parameters of unsaturated soil. Mair et al. [16] used *ACO* algorithms to find a near-optimal solution to a water distribution system. They discussed that the *ACO* algorithms may form an attractive alternative to $GA_s$ for the optimal design of water distribution systems. Abbasi et al. [1] used *ACO* algorithm to optimally design a water conveyance system under steady state condition with uniform and predefined discretization schemes.

Although *ACOs* have basically been presented to solve the problems with discrete search spaces, discretization process was proposed later in favor of continuous domains. Jalali et al. [17] employed the concept of multi ant colonies to randomly discretize the continuous search space. The method was successfully applied for operation of a multi-reservoir system. On the other side, many researchers have attempted to extend the concept of the basic ant algorithms into continuous domains. Bilchev and Parmee [18] proposed Continuous *ACO* (*CACO*) as the first method directly using *ACO* in continuous search space. Later studies resulted in Asynchronous Parallel Implementation (*API*) algorithm [19], Continuous Interacting Ant Colony (*CIAC*) [20] and $ACO_R$ [21]. Afshar [22] proposed and applied a so-called parameter free continuous ant colony optimization algorithm for the optimal design of storm sewer network under constrained and unconstrained approaches. Recently, Madadgar and Afshar [23] introduced the improved $ACO_R$ in which the *adaptation operator* and *explorer ants* are employed to promote the search results. The performance of improved version was satisfactory on a set of mathematical functions and also the optimal operation of a hydropower reservoir problem.

In contrast to continuous spaces, very limited studies have attempted to solve the mixed-variable problems (with both continuous and discrete decision variables) by means of ant methods. In a recent study, Schlüter et al. [24] extended continuous ant approach to mixed integer search space. Their algorithm, $ACO_{mi}$, was implemented on mathematical benchmark problems and then on a load-bearing thermal insulation system as an engineering problem.

This paper develops and compares the performance of a number of ant-based approaches for mixed-variable problems. The proposed algorithms are initially tested on three mathematical problems and then applied to a mixed-variable problem in water resources engineering. The optimum design of a large scale irrigation main and inter basin water transfer pipeline is considered and the performance of introduced methods are compared.

## 2. ANT COLONY ALGORITHMS; AN OVERVIEW

An interesting and very important behavior of ant colonies is their foraging behavior, and in particular, their ability to find the shortest route between their nest and a food source. The path taken by individual ants from the nest to the food source is essentially random [10]. However, when traveling, ants deposit a substance called pheromone, forming a pheromone trail as an indirect means of communication. As more ants choose a path to follow, the pheromone on the path builds up, making it more attractive for other ants to follow.

In the $ACO$ algorithm, artificial ants are permitted to release pheromone while developing a solution or after a solution has been fully developed, or both. The amount of pheromone deposited is made proportional to the goodness of the solution an artificial ant develops. Rapid drift of all ants toward the same part of the search space is avoided by employing the stochastic component of the choice decision policy and the numerous mechanisms such as pheromone evaporation, explorer ants, and local search.

In order to successfully apply the $ACO$ algorithms to combinatorial optimization problems, it is recommended to project the problem on a graph. Consider a graph ***G****= (****D, L, C****)*, in which ***D, L***, and ***C*** are the sets of decision points, options and costs associated with option ***L***, respectively. A feasible path on the graph is called a solution and the path with minimum cost is called the optimum solution.

The transition rule used in the original ant system is defined as [10]:

$$P_{ij}\ (k,t) = \begin{cases} \dfrac{[t_{ij}(t)]^{a}\ [h_{ij}]^{b}}{\sum\limits_{j=1}^{J} [t_{ij}(t)]^{a}\ [h_{ij}]^{b}} & if \quad j \in N_k(i) \\ 0 & otherwise \end{cases} \tag{1}$$

where $P_{ij}(k,t)$ is the probability that ant k selects option $l_{ij}$ for decision point $i$ at iteration $t$; $t_{ij}(t)$ is the concentration of pheromone on arc ($i$, $j$) at iteration $t$; $h_{ij} = \dfrac{1}{c_{ij}}$ is the heuristic value representing the cost of choosing option $j$ at decision point $i$; $N_k$ ($i$) is the feasible neighborhood of ant $k$ when located at decision point $i$; and $a$ and $b$ are two parameters that control the relative importance of the pheromone trail and heuristic value. The heuristic value $h_{ij}$ is analogous to providing the ants with sight and is sometimes called visibility. This value, in static problems, is calculated once at the start of the algorithm and remains unchanged during the computation process.

To simulate pheromone evaporation, the pheromone evaporation coefficient $(r)$ is defined which enables greater exploration of the search space and minimizes the chance of premature convergence to sub-optimal solutions upon completion of a tour by all ants in the colony. The global trail updating is done as follows:

$$t_{ij}(t+1) = (1-r).t_{ij}(t) + r.\Delta t_{ij}(t) \tag{2}$$

Where $t_{ij}(t+1)$ is the amount of pheromone trail on option $j$ of the $i$th decision point at iteration $t+1$; $0 \le r \le 1$ is the coefficient representing the pheromone evaporation and $\Delta t_{ij}(t)$ is the change in pheromone concentration associated with arc $(i, j)$ at iteration $t$. The amount of pheromone $t_{ij}(t)$ associated with arc $(i, j)$ is intended to represent the learned desirability of choosing option $j$ when at decision point $i$.

Various methods have been suggested for calculating the pheromone changes. The method used here was originally suggested by Dorigo and Gambardella [25] in which only the ant which produced the globally best (gb) solution from the beginning of the trail is allowed to contribute to pheromone change:

$$\Delta t_{ij}(t) = \begin{cases} 1/G^{k^*_{gb}} & if \quad (i, j) \in \text{tour done by ant } k^*_{gb} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Where $G^{k^*_{gb}}$ is value of the objective function for ant $k^*_{gb}$, which is the ant with the best performance within previous iterations.

Satisfactory performance of *ACOs* in discrete search spaces persuaded the researchers to develop ant-based algorithms for continuous search spaces. Initial attempts suggested the conversion of original continuous search space into a discrete one. The discrete domain is specified by a finite set of allowable options at any construction steps. As the continuous domain is replaced by a discrete one, the resulting solutions may lose some of the potentially good solutions according to the size of the finite sets. Since the coarse discrete space leads to worse solutions than a fine one, there would be a tendency to generate more populous discrete domains.

Jalali et al. [17] proposed a new multi-colony method which utilizes a multi-colony system to properly limit the searching process into the high-quality regions. Different colonies with heterogeneous discrete patterns help to provide a non-homogeneous and dynamic discrete domain.

Recently, probability density functions (pdf$_s$) have been used to generate real values for continuous variables. The approach seems efficient to search through continuous spaces as it almost covers the whole domain and assigns a probability value to any points of the space. Since the assigned probability values (via pdf) differ from point to point, it is more convenient to model the variations of pheromone amount along the search space. In other words, the area around peak point of pdf receives more chance to be selected rather than those around the tail.

It means that the pdf provides the capacity to search entire decision space considering the varieties in attraction of different areas. In order to simulate the pheromone model, a single pdf may be used to represent the probabilities of different parts in search space. The proposed pdf at $i$ th construction step may be defined as [26]:

$$t(x) = e^{-\frac{(x - x_{\min})^2}{2s^2}} \tag{4}$$

Where $x_{\min}$ and $s$ define the mean and standard deviation of the pdf associated to the $i$ th construction step, respectively.

The mean value of the distribution at $i$ th step is equal to the decision value generated by the global best ant up to the current iteration. The variance of the pdf describes the concentration of ants around the best found solution and is defined as follows [26]:

$$s^2 = \frac{\sum_{j=1}^{k} \frac{1}{f_j - f_{\min}} \left(x_j - x_{\min}\right)^2}{\sum_{j=1}^{k} \frac{1}{f_j - f_{\min}}} \tag{5}$$

Where, $f_{best}$ and $f_j$ are the fitness values associated to the best and the $j$ th ants in the population, respectively; and $k$ is the population size. As is clear, the concentration of ants around the best solution determines the standard deviation of pdf. In any iteration, ants generate the continuous values according to Equations 4 and 5 except in the first iteration, while all ants produce completely random values. Since a single pdf has only one peak point, the uni-pdf method is not capable to simulate the areas with multi-promising regions. To overcome this shortcoming, the Gaussian kernel pdf (Figure 1), which is weighted sum of several individual Gaussian functions $g_l^i(x)$, has been suggested [21]**:**

$$G^i(x) = \sum_{l=1}^{k} w_l g_l^i(x) = \sum_{l=1}^{k} w_l \frac{1}{s_l^i \sqrt{2p}} e^{-\frac{\left(x - m_l^i\right)^2}{2 s_l^{i\,2}}} \tag{6}$$

where $k$ is the number of single pdfs making the Gaussian kernel pdf at $i$ th construction step; $w$, $m^i$ and $s^i$ are the vectors of size $k$ define weights, means and standard deviations of individual Gaussian functions at $i$ th construction step, respectively.

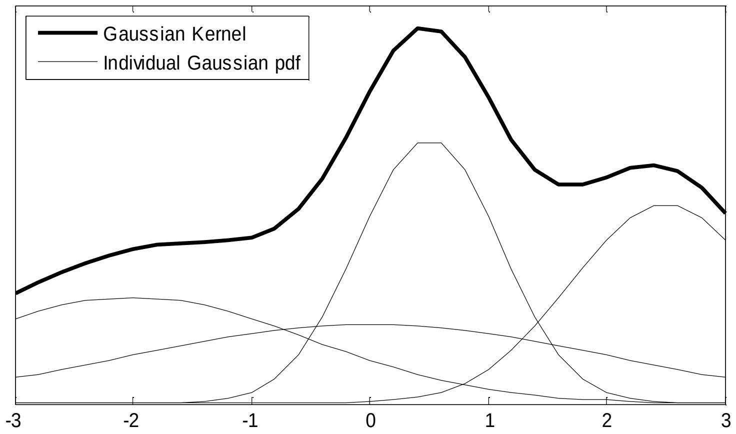


Figure 1. Example of a Gaussian kernel pdf including 4 Gaussian functions on the interval (–3, 3) (adopted from Madadgar and Afshar [23])

## 3. PROPOSED ACO-BASED METHODS FOR MIXED-VARIABLE PROBLEMS

In contrast to discrete and/or continuous search spaces, few studies have already tried to extend ant based approaches to mixed variable problems [24]. This article attempts to propose and structure four different ant-based algorithms (methods A-D) for mixed-variable domains. Each method combines different ant algorithms developed for either discrete and/or continuous search spaces, except Method D which is inherently proposed for mixed-variable problems.

*Method A*

The first algorithm combines multi-colony ant approach of Jalali et al. [17] with improved version of ant system [27]. The multi-colony ant approach is used to develop a heterogeneous and dynamic discrete scheme into the original continuous space which will lead the searching process toward the high quality regions of continuous space.

Figure 2 illustrates a three-colony system when the information exchange process is executed. As shown, each colony has its own discrete scheme and the agents conduct the search process in the consequent discrete space. The information exchange technique is also beneficial to concentrate finely around the best solution. Once the information exchange criterion is satisfied, the best solution of any colonies is determined. Afterwards, the search space for any decision variable at each colony is limited into the maximum and minimum values obtained by the best solutions of all the colonies. Then, the new search space, which becomes obviously smaller, is transformed into discrete space with regard to the specified pattern of each colony. As is clear, searching via multi colonies, including non-homogeneous discrete spaces, provides broad and fine exploration in continuous domains.

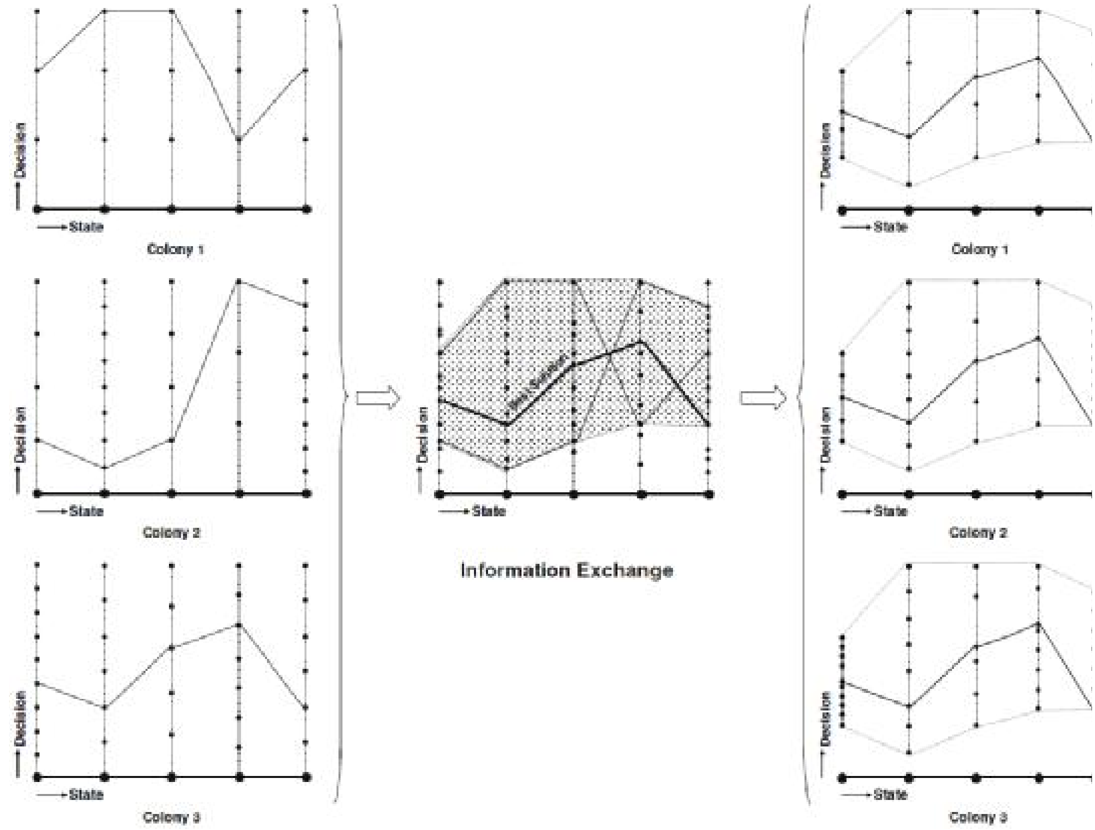


Figure 2. Example of a three-colony system in multi-colony approach a) before, b) through, c) after information exchange process (Adopted from Jalali et al. [17])

To accomplish search process through discrete pattern, regardless to the variable type (either discrete or continuous), the improved version of ant system [27] is considered. Since original $ACO$ algorithms suffer from the premature convergence to the local optima, or stagnation point, some studies have been conducted to troubleshoot such undesirability. Jalali et al. [28] employed a combination of explorer ants, local search, and the pheromone promotion (*PP*) techniques to minimize the possibility of the premature convergence syndrome, however, the results of 10 runs with different seeds revealed relatively high standard deviation, which may be considered as an index of results diversity. Therefore, Jalali and Afshar [27] introduced pheromone re-initiation (*PRI*) and partial path replacement (*PPR*) mechanisms to further reduce the possibility of premature convergence. Thus, the improved version of ant system benefits from *PP*, *PRI*, and *PPR* techniques in order to attain a satisfactory performance of $AS$ algorithm.

*Method B*

The second proposed algorithm combines the uni-pdf ant approach [26] with modified ant system [28] regarding to construct, respectively, continuous and discrete values for associated variables. The considered modified ant system utilizes explorer ants, local search, and the pheromone promotion $(PP)$ mechanism as the major means to prevent falling in local optima.

*Method C*

Deficiency of single pdf function in continuous search spaces supports the application of Gaussian kernel pdf which prepares efficient pheromone modeling of a multi-promising area. Substituting the single pdf function with Gaussian kernel pdf forms the main difference between the second and third developed method for mixed domains. For discrete decision variables both the second and third methods use the modified ant system [28].

The Gaussian kernel pdf is a weighted sum of several individual pdf functions (Eq. 6). Defining the means and standard deviations of individual Gaussian functions at $i$ th construction step is executed due to an updatable archive. The archive $T$ stores a certain number ($k$) of best solutions. In an n-dimensional problem, the archive keeps the values of $n$ variables associated with any selected solution $s_1$ . Pheromone updating is accomplished by adding the set of new superior solutions to the solution archive and in contrary, removing the same number of inferior solutions in order to keep the total size of the archive unchanged.

The shape of the Gaussian kernel pdfs are described by the vectors $w$ , $m^i$ , and $s^i$ which are determined by the solutions of archive.

At $i$ th construction step, the vector $m^i$ , associated to the Gaussian kernel pdf ($G^i$), is made of the values of $i$ th variable of the solutions in the archive.

After a complete iteration, all solutions (either in the archive or currently constructed) are ranked according to their fitness value. In the next step, $k$ superior solutions fill up the archive and the rests will be omitted (i.e., solution $s_l$ has rank $l$).

The weight $w_l$ of solution $s_l$ is calculated by a Gaussian function as [21]:

$$w_l = \frac{1}{qk\sqrt{2p}} e^{-\frac{(l-1)^2}{2q^2k^2}} \tag{7}$$

$q$ $(>0)$ is a tuning parameter, the value of which is highly affect the convergence rate of algorithm. Smaller the values, more fast the convergence speed and vice versa.

To prepare easy sampling from Gaussian kernel pdf, each ant starts the construction process by choosing exactly one solution of archive. Then it uses the Gaussian functions associated with the chosen solution for all $n$ construction steps. The probability of choosing ranked solution $l$ (and consequently Gaussian functions $l$) is proportional to the weight of Gaussian function $l$ (Eq. 7)

For step $i$, $s_l^i$ is calculated as [21]:

$$s_l^i = x \sum_{e=1}^{k} \frac{\left| s_e^i - s_l^i \right|}{k-1} \tag{8}$$

Where $x(>0)$ is a parameter which behaves similar to the coefficient of pheromone evaporation in $ACO$.

*Method D*

Madadgar and Afshar [23] suggested an adaptive process to adjust the value of parameter $q$ during the successive iterations in order to conduct an efficient searching through decision space. The adaptation on $q$ helped the method to effectively benefit from both exploration and exploitation concepts. They also employed the explorer ants to execute finely local search. These special ants minimize the chance of trapping in local optimums and premature convergences. They proposed a new ant model, which is adapted to tackle the problems with mixed-variable search space. This model is based on the improved version of continuous ant algorithm [23] and is considered here as the fourth method. The model initiates the solution construction process through the genuine or virtual continuous spaces. In other words, the model deals with all variables as continuous ones. Afterward, the practicable values of discrete variables are gained regarding to a transformation from associated virtual continuous space into the original discrete one. The performance of the algorithm was tested on three mathematical functions and a forced water main design problem.

Table 1 illustrates a quick review on the proposed and/or nominated methods and their approaches in treating continuous and discrete variables.

Table 1. Identification of four ant-based methods proposed for mixed variable problems

| | **Applied Algorithms** | |
|---|---|---|
| | **Discrete Variables** | **Continuous Variables** |
| Method A | $ACS_{gb}$-PP-PRI-PPR (Jalali and Afshar [27]) | Multi-Colony model (Jalali et al. [17]) |
| Method B | $ACS_{gb}$-PP (Jalali et al. [28]) | Uni-pdf model (Pourtakdoust and Nobahari [26]) |
| Method C | $ACS_{gb}$-PP (Jalali et al. [28]) | Multi-pdf model (Madadgar and Afshar [23]) |
| Method D | Multi-pdf model (Madadgar and Afshar [23]) | Multi-pdf model (Madadgar and Afshar [23]) |

## 4. APPLICATION OF METHODS

This section tests and compares the performances of the methods on a set of mathematical test functions and a forced water main design problem. The section is followed by profound discussion on obtained results.

### *4.1. Test functions*

A set of three mathematical functions with mixed-variable domains are employed to survey the performance of presented four ant-based methods in mixed-variable problems. Table 2 lists the functions along with according global optimums. Test functions are taken from Yiqing et al. [29]. These complicated functions have already been solved by some other evolutionary algorithms ([30], [29]), and hence, a comprehensive judge on the performance of applied algorithms may be attained by comparison between the results.

Table 2. Summary of test functions

| Problem | Mathematical Formulation | Global Optimum |
|---|---|---|
| 1 | $Min\ \ f = 2x_1 + x_2 - y$<br>$Subject\ \ to:$<br>$x_1 - 2e^{-x_2} = 0$<br>$-x_1 + x_2 + y \le 0$<br>$0.5 \le x_1 \le 1.4$<br>$y \in \{0,1\}$ | $f(x_1, x_2, y) =$<br>$f(1.375, 0.375, 1) = 2.124$ |
| 2 | $Min\ \ f = 7.5y_1 + 5.5y_2 + 7u_1 + 6u_2 + 5x$<br>$Subject\ to:$<br>$y_1 + y_2 = 1$<br>$z_1 = 0.9x_1\left(1 - e^{-0.5u_1}\right)$<br>$z_2 = 0.8x_2\left(1 - e^{-0.4u_2}\right)$<br>$z_1 + z_2 = 10$<br>$x_1 + x_2 = x$<br>$u_1 \le 10y_1\ ,\ u_2 \le 10y_2$<br>$x_1 \le 20y_1\ , x_2 \le 20y_2$<br>$x_1, x_2, z_1, z_2, u_1, u_2 \ge 0 \quad , \quad y \in \{0,1\}$ | $f(x, y_1, y_2, u_1, u_2) =$<br>$f(13.42799, 1, 0, 3.514237, 0) = 99.23963$ |
| 3 | $Min\ \ f = (y_1 - 1)^2 + (y_2 - 2)^2 + (y_3 - 1)^2 - \ln(y_4 + 1)$<br>$+ (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2$<br>$Subject\ to:$<br>$y_1 + y_2 + y_3 + x_1 + x_2 + x_3 \le 5$<br>$y_3^2 + x_1^2 + x_2^2 + x_3^2 \le 5.5$<br>$y_1 + x_1 \le 1.2$<br>$y_2 + x_2 \le 1.8$<br>$y_3 + x_3 \le 2.5$<br>$y_4 + x_1 \le 1.2$<br>$y_2^2 + x_2^2 \le 1.64$<br>$y_3^2 + x_3^2 \le 4.25$<br>$y_2^2 + x_3^2 \le 4.64$<br>$x_1, x_2, x_3 \ge 0 \quad , \quad y_1, y_2, y_3, y_4 \in \{0,1\}$ | $f(x_1, x_2, x_3, y_1, y_2, y_3, y_4) =$<br>$f(0.2, 0.8, 1.907878, 1, 1, 0, 1) = 4.579582$ |

To have a sound judge, a certain degree of convergence is determined as stop criterion for all four methods. This is the same criterion that has been applied in other researches [29]:

$$\left| f_k - f_{k-\Delta k} \right| < 10^{-5} \quad , \quad \forall\, k > \Delta k \tag{9}$$

In which, $f_k, f_{k-\Delta k}$ are the objective values of the best-found solutions within any consecutive $k^{th}$ and $(k-\Delta k)^{th}$ iterations. The iteration interval is assumed $\Delta k = 50$. The termination criterion indicates that if the objective values of the best solutions in consecutive iterations with an interval of $\Delta k = 50$ remains close enough to each other, the desirable convergence of the employed algorithm is assumed to be achieved.

Table 3 briefly shows the performance of proposed methods on test functions. For each test function and employed method, the mean number of function evaluations and the percentage of successful executions over all independent runs are reported. Successful executions are those converge to the optimal solution of considered problem. The reported values are obtained after 100 independent executions.

Table 3. Performances of methods A-D on mathematical test functions

| **Problem** | **Method A** | **Method B** | **Method C** | **Method D** |
|---|---|---|---|---|
| 1 | 9000/100 | 1648/80 | 781/95 | 576/100 |
| 2 | 240000/90 | 1963/96 | 1407/90 | 763/100 |
| 3 | 75000/100 | 2437/100 | 1264/100 | 761/97 |

In other studies ([30], [29]), some non ant-based methods have been employed to solve these test functions, and the reported results are presented in Table 4. These results are also based on 100 independent runs of applied optimization methods.

Table 4. Performances of non ant-based optimization methods on test functions

| **Problem** | **GA [1]** | **M-SIMPSA [1]** | **Original PSO [2]** | **R-PSO_c [2]** |
|---|---|---|---|---|
| 1 | 13939/100 | 14440/100 | –[3] | 3500/100 |
| 2 | 22489/100 | 42295/100 | –[3] | 4000/100 |
| 3 | 102778/60 | 63751/97 | 30000/80 | 30000/100 |

**[1] Costa and Olivera [30]**
**[2] Yiqing et al. [29]**
**[3] All executions were unsuccessful**

Among ant-based methods, the last method (method D) performs much better than others, and methods B and C are appeared competitive approaches. This may partially be due to approach taken by methods B-D to treat continuous variables. Methods B-D all employ the continuous-natured ant approaches in dealing with continuous domains. Method B benefits from uni-pdf approach [26], and methods C and D uses the improved multi-pdf method [23]. However, method D achieved the termination criteria in remarkably less number of function evaluations than other algorithms. This may rise from the sampling the search spaces regardless the variable type which may cause model D to proceed in both domains with rather equal paces. Note that the improved version of $ACO_R$ [23], as the main core of method D, is a potent method, by itself, in continuous optimization problems. Method A converges to global optimum in most of executions but within great number of function evaluations in comparison with the rest three methods. It treats the continuous variables through discrete-scheme domains. Probably, this causes method A to converge towards optimum solutions in quite low rate.

The results obtained by some non ant-based methods in Table 4 indicate the remarkable performance of applied ant approaches in locating near optimal solutions. The only competitive method in Table 4 is R-PSO_c [29] which could satisfy the stop criterion in rather small number of function evaluations. It performs better than Method A in Table 3, as well. As shown, Original PSO could not successfully converge to global optimums in an even one execution over 100 runs in problems 1 and 2.

Consequently, the proposed ant-based approaches are investigated as efficient methods for considered test functions in mixed variable domain. To further illustrate the performance of proposed methods as alternative approaches, it is strongly suggested to apply these methods to complex optimization problems specifically in engineering studies.

### *4.2. Water main design problem*

To acquire a comprehensive perception of proposed methods' performance, a mixed-variable optimization problem in water engineering, a forced water main design problem, is employed and discussed in following sections. The problem includes a pipeline system with pre-defined layout, nodes, and segment lengths. Constant flow of water is supposed to be conveyed through the nodes; while the piezometric head at all nodes must be continuously kept above a minimum requirement. The nodes are possible to be assigned pump stations. The pipe diameters and pumping heads in potential stations are to be determined through an optimization model. In other words, the model is to find the best combination of pipe diameters and pumping heads through the system. Model's objective is defined as minimum total annual cost including the initial investment and operational costs of the conveyance system (Appendix):

$$Min \;\; Z = \sum_{n=1}^{NN} C_n(hp_n) + \sum_{i=1}^{NR} C_i(D_i) \tag{10}$$

In which, $C_n(hp_n)$ and $C_i(D_i)$ are the cost functions associated to pumping head at node $n$ and pipe diameter in segment $i$, respectively. $hp_n$ is the pumping head at node $n$; $D_i$ is the pipe diameter in segment $i$ ; $NN$ is the number of nodes; and $NR$ is the number of reaches (segments). Certainly, the number of nodes $(NN)$ is equal to $NR+1$.

The constraints of the model are presented as follow:

$$V_{\min} \le V_i \le V_{\max} \qquad\qquad i = 1,\ldots,NR \tag{11}$$

$$D_{\min} \le D_i \le D_{\max} \qquad\qquad i = 1,\ldots,NR \tag{12}$$

$$h_{\min} \le h_n \le h_{\max} \qquad n = 1,\ldots,NN \tag{13}$$

Where, $V_i$ is water velocity at reach $i$; $h_n$ is the piezometric head at node $n$; and $V_{\min}, V_{\max}, h_{\min}, h_{\max}$ are the minimum and maximum allowable velocities in all reaches and minimum and maximum allowable piezometric head at all nodes, respectively. Since the pipe diameter in each segment and according flow velocity are inherently correlated via water flow $(Q)$ equation, it is applicable to determine the allowable set of pipe diameters regarding to the allowable range of velocities for a given water flow $(Q)$ in the system. If so, the velocities will be implicitly fallen within allowable range, and the only remain constraints will be those on pipe diameters (Eq. 12) and piezometric heads (Eq. 13).

Piezometric head at node $n$ may be defined by energy equation as follow:

$$h_n + \frac{V_n^2}{2g} + \left(h_p\right)_n = h_{n+1} + \frac{V_{n+1}^2}{2g} + \left(h_{Loss}\right)_i \qquad n = 1,\ldots,NN-1 \tag{14}$$

Where, $h$ is the piezometric head; $h_p$ is the pumping head; $h_{Loss}$ is the total head loss between two points including both local and friction losses; and the indexes $n$ and $n+1$ refer to the beginning and ending nodes of link $i$. Since the velocities are rather small in the long pipelines, the terms $V_n$ , $V_{n+1}$ are negligible.

Determination of the energy loss between two points requires accounting both friction and local losses. However, in long pipes, one may reasonably disregard the local ones in energy loss calculation. In fact, the energy loss in long pipes is significantly pronounced to friction loss rather than local loss. Hence, $h_{Loss}$ in Eq. 14 is expressed by following Hazen-Williams equation for friction loss:

$$\left(h_f\right)_i = 10.7 \times L_i \times \left(\frac{V_i \times D_i}{C_H}\right)^{1.852} \times \frac{1}{D_i^{4.87}} \qquad i = 1,\ldots,NR \tag{15}$$

In which, $h_f$ is the friction loss; $L$ is the pipe length; and $C_H$ is the Hazen-Williams coefficient.

This paper considers a water main system with a pre-defined layout as depicted in Figure 3. The main path consists of 18 nodes and 17 reaches with lengths listed in Table 5. Water flow is assumed constantly $0.3\left(m^3/s\right)$ through whole conveyance path, and the Hazen-Williams coefficient is equal to $C_H = 120$ . The velocity in any reaches, pressure head, and pumping head in any nodes may fluctuate in allowable ranges of $[0.4, 2.6]\,m/s$, $[3,150]\,m$, and $[3,80]\,m$, respectively. According to water discharge and permitted range for water velocity in the system, the pipe diameters ought to fall within the range of $[0.4, 0.8]\,m$. However, regarding to available diameters in market, the according continuous range is divided into discrete measures with 0.05m intervals.

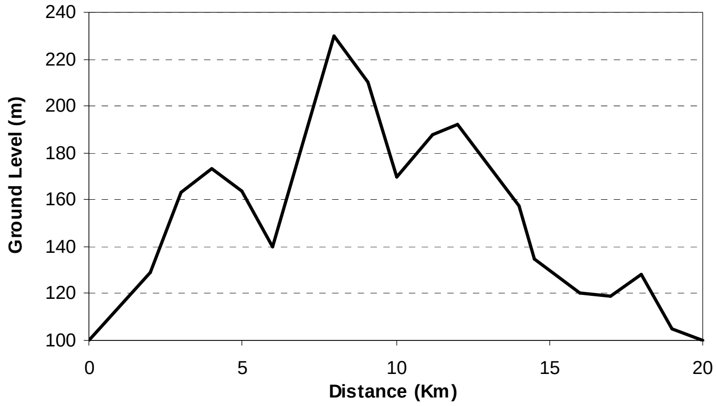


Figure 3. Pipeline layout of the case study

Table 5. Lengths of segments

| **Reach number** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
|---|---|---|---|---|---|---|---|---|---|
| Length (m) | 2000 | 1000 | 1000 | 1000 | 1000 | 2000 | 1100 | 900 | 1200 |
| Reach number | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | |
| Length (m) | 800 | 2000 | 500 | 1500 | 1000 | 1000 | 1000 | 1000 | |

Inclusion of 18 nodes and 17 reaches in the system leads the model to an optimization problem with 17 discrete decision variables (pipe diameter in each reach) and 18 potential continuous decision variables (pumping heads at each node). Each node in the system may possibly be assigned a pump station, and if so, the pumping head is accordingly determined. Methods A to D employ different approaches for recognition of pump stations at different nodes. Method A adds a pumping head of zero to each decision step (node), and if it is chosen by an agent, the absence of a pump station at according node is indicated. Method B suggests a quite different approach which distinguishes the absence of pump station just after determination of the pumping head. If chosen value for pumping head is fallen beneath its allowable range, the absence of pump station at according node is discerned. Otherwise, a pump station with chosen pumping head is allocated to the node. Methods C and D employ a same approach towards the matter. Using Method D, two approaches can be provided for making decision on pump stations presence. In first approach, a binary variable can be attributed to each node. The value of binary variable shows the presence or absence of a pump station at according node. Hence, this approach doubles the number of discrete decision variables and may exceed the computational effort in large-scale problems. The second approach refers to the main core of method D which employs the Gaussian pdfs from solution archive for generating continuous values. This approach introduces and inserts a new solution called *zero solution* to the solution archive in parts due to pumping heads. Choosing zero value for the zero solution at any node implies the absence of pump station. If an agent chooses a solution other than zero solution, the presence of pump station is indicated and the generated value from chosen pdf will represents the pumping head at according node. This approach subtly recognizes the presence of pump stations at the nodes without insertion of any extra array of decision variables. Then, this study applies the second approach for method D.

Once an agent makes a solution, it should be checked whether all constraints including those on water velocity in pipes and those on piezometric heads at nodes have been satisfied. If not, the associated agent must be penalized. Since the allowable range of pipe diameters are determined such that the permitted range of velocity is automatically satisfied, the only remain applicable constraints are those on piezometric heads. The following penalty function expresses how an agent will be penalized if its decisions lead to violation of permitted range of piezometric head at any node:

$$Penalty_n = \begin{cases} PF \times (h_{\max} - h_n)^2 & if\ h_n > h_{\max} \\ PF \times (h_{\min} - h_n)^2 & if\ h_n < h_{\min} \end{cases} \quad n = 1,...,NN \tag{16}$$

In which, $PF$ is the penalty factor indicating the severity of violating the piezometric head constraint. Since a minimization problem is conducted, the objective value of a penalized agent should be increased as follow:

$$Z = \sum_{n=1}^{NN} C_n(hp_n) + \sum_{i=1}^{NR} C_i(D_i) + \sum_{n=1}^{NN} Penalty_n \tag{17}$$

This way, an agent is degraded if it generates a solution in non-feasible space. Hence, after a while, the population is spontaneously encouraged to search through feasible domain.

### *4.3. Results and discussion*

As stated, the problem consists of 35 decision variables from which 17 pipe diameters are discrete and remaining 18 variables are in continuous domain (pumping heads). Discrete decision space on pipe diameter is defined on the range of $[0.4, 0.8]$ meter with 0.05$m$ intervals, and the pumping heads have the allowable range between 3 and 80 meters. In following, the results of proposed methods will be reported and discussed. To have a comparison of obtained results, the problem is also solved by a potent nonlinear solver, Lingo 9.0. To figure out the presence of pump stations at the nodes, the mathematical model lends itself to a Mix Integer Non-Linear Programming $(MINLP)$ problem. It reports two supreme local optimums for the problem with different composition of pumping stations and pipe diameters but with the same objective value of 122.57 thousands dollar as total annual cost. Both optimum solutions locates a pair of pump stations through the pipeline path: one on the nodes number 1, 3 and another one on the nodes number 1, 4. The former optimum solution reports the pumping head of 68.42m for each station, and the latter reports the values of 77.92m and 58.92m as designate pumping heads. These reported optimums follow the same pattern for pipe diameters. That is, the only difference in theses optimums is derived from pump stations, in both terms of locations and pumping heads.

Table 5 presents the results obtained by proposed methods after 20 executions. Each algorithm conducts the search process through pre-defined number of iterations as listed in Table 6. Most crucial specifications of the algorithms are summarized in column 2. Numbers of function evaluations leading to reported values in columns 4-7 are listed in column 3. Columns 4 and 6 show the objective values of the best and worst found solution after 20 independent runs of each algorithm. Values in parenthesis are the number of pump stations that according solution assigns to the pipeline. For instance, the best and worst solutions found by method A allocate, respectively, two and five pump stations to the system. The mean and standard deviation of obtained objective values are reported in columns 5 and 7.

Table 6. Results found by proposed methods

| Method | Specifications | NFE[1] | Best | Mean | Worst | S.D. |
|---|---|---|---|---|---|---|
| A | No. of colonies:3<br>Population size: 150<br>Total ants: 450<br>No. of iterations: 2000 | 900,000 | 127.18(2) | 134.06 | 143.32(5) | 4.98 |
| B | Population size: 30<br>No. of iterations: 500 | 15,000 | 123.26(2) | 125.45 | 133.32(3) | 2.97 |
| C | Population size: 30<br>No. of iterations: 500 | 15,000 | 122.91(2) | 125.39 | 130.67(3) | 2.89 |
| D | Population size: 20<br>No. of iterations: 300 | 6,000 | 122.71 (2) | 123.12 | 124.38 (2) | 0.41 |

[1]**Number of Function Evaluations**

Method D located near the optimal solution in remarkably less number of function evaluations than other three methods. Its best found solution locates quite close to the optimum solutions (Table 7). As seen, the system components including pipe diameters and pumping heads due to the best found solution by method D follow almost the same pattern as those due to reported local optimums. From Table 6, the mean and standard deviation of objective values over 20 runs indicate the robustness of method D in locating near optimum solution specifically in comparison with other methods.

The next superior results are respectively obtained by methods C, B, and A. Methods B and C perform quite close to each other which may rise from the same approaches these methods use to make decisions on each variable type. Both methods employ the discrete-based ant models for discrete decision variables; and in continuous space, they utilize the continuous-based approaches (Table 1). Method B employs the uni-pdf model while method C uses multi-pdf approach. Hence, these algorithms advance towards rather same promising areas with almost the same convergence rates. In both methods, a population of 30 ants explores the decision space thorough 500 iteration which is equivalent to total 15,000 function evaluations (Table 6). In contrast, method A suggests the same approach to tackle any types of decision variables. It utilizes the discrete ant models for both types of discrete and continuous decision variables. Treating the continuous decision variables as they are in discrete domain may cause the weak performance of method A in comparison with other proposed methods.

However, results obtained by method A provide decision makers with a variety of solutions including 2 to 5 pump stations through the pipeline but with rather different objective values. In addition, it required much more number of function evaluations to find the reported results.

Table 7. Comparison between the reported local optimums and the best found solution by method D

| | **Reported Local optimums** | **Best found solution** |
|---|---|---|
| **Pipe no.** | **Pipe diameter (m)** | **Pipe diameter (m)** |
| 1 | 0.8 | 0.8 |
| 2 | 0.8 | 0.8 |
| 3 | 0.8 | 0.8 |
| 4 | 0.8 | 0.75 |
| 5 | 0.8 | 0.8 |
| 6 | 0.8 | 0.8 |
| 7 | 0.45 | 0.45 |
| 8 | 0.45 | 0.5 |
| 9 | 0.45 | 0.45 |
| 10 | 0.45 | 0.45 |
| 11 | 0.4 | 0.4 |
| 12 | 0.45 | 0.45 |
| 13 | 0.4 | 0.4 |
| 14 | 0.45 | 0.4 |
| 15 | 0.45 | 0.5 |
| 16 | 0.4 | 0.4 |
| 17 | 0.4 | 0.4 |
| Pumping Head (m) | Node 1: 68.42<br>Node 3: 68.42<br>Or<br>Node 1: 77.92<br>Node 4: 58.92 | Node 1: 70.05<br>Node 3: 66.98 |

Figure 4 compares the ground level with the energy grade line for the best solutions found by method D. It assigns two pump stations at node numbers 1 and 3 with respective pumping heads of 70.05 and 66.98 meters. As is obvious, the energy grade line is adequately located above the ground level through the path and depicted jumps in energy grade line are occurred in nodes with pump stations.

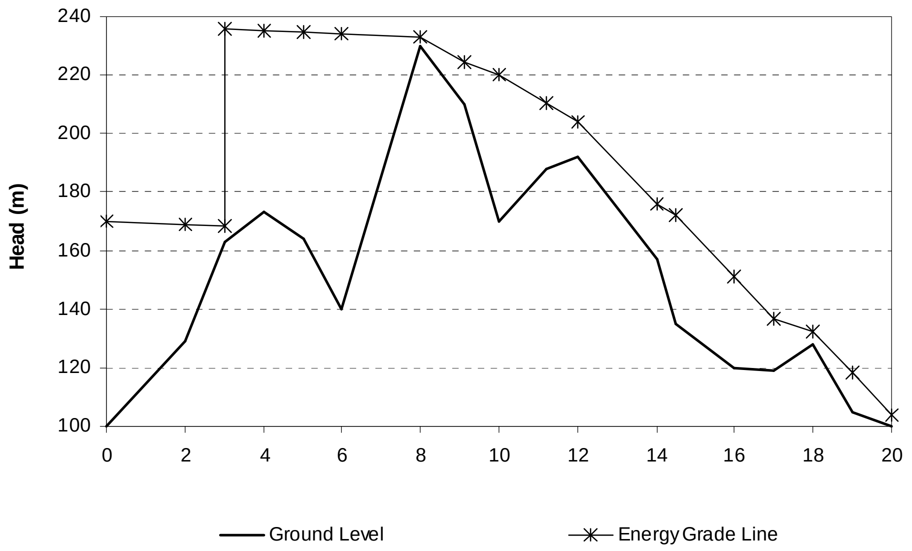


Figure 4. Energy grade line due to the best found solution by method D

## 5. CONCLUDING REMARKS

Ant colony algorithm was basically introduced to solve the optimization problems with discrete decision variables. Efficient performance of *ACOs* in discrete domains led to different adjustments which enable the ant models to be extended into the continuous-variable problems. Despite large number of optimization problems in mixed-variable domains which include both discrete and continuous decision variables, lack of ant-based algorithms in this area of research is quite sensible. This paper proposed four ant-based approaches to solve the mixed variable problems. Each of them is supported by potent ant models in continuous and/or discrete domains. To compare the performances of the methods, they were applied first to a set of mathematical test functions and then to an engineering design problem. Test functions were relatively elaborate in constraints, and the employed engineering problem is due to a highly non-linear forced water main design. The water main problem is to find the optimum combination of pipe diameters and pumping heads in a pre-defined pipeline path as to achieve the minimum annual cost of the system. Pipe diameters form the set of discrete variables and pumping heads are regarded as continuous variables. Results demonstrate the satisfactory performance of proposed methods on considered mathematical and engineering problems; however, further application to mixed-variable domains is highly suggested. These methods may be rationally noted as alternative approaches in science and engineering optimization studies.

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## APPENDIX

Total cost of the assumed water main includes initial investments and annual operation costs.

- The initial investments encompass the costs on:
- purchase and installation of the pumps ($Cost_p$)
- pump station house ($Cost_s$)
- accessory equipments ($Cost_{eq}$)
- electrical instruments ($Cost_{el}$)
- purchase and fixing the pipes which is dependent on the pipes' diameters ($Cost_d$)

The annual operation cost is due to the required electricity for pumping the water ($Cost_e$).
The noted costs, at each node or reach, are expressed as follows:

$$\begin{aligned} &Cost_p = Q \times (a_p + b_p \times h_p + c_p \times h_p^2 + d_p \times h_p^3) \\ &a_p = 21225.6 \\ &b_p = 462.368 \\ &c_p = -1.1895 \\ &d_p = 6.2944 \times 10^{-3} \end{aligned} \tag{18}$$

$$\begin{aligned} &Cost_s = a_s + b_s \times Cost_p + c_s \times Cost_p^2 + d_s \times Cost_p^3 \\ &a_s = 4461.41 \\ &b_s = 0.483496 \\ &c_s = 1.28 \times 10^{-7} \\ &d_s = -7.85 \times 10^{-15} \end{aligned} \tag{19}$$

$$\begin{aligned} &Cost_{eq} = a_{eq} + b_{eq} \times Cost_p + c_{eq} \times Cost_p^2 + d_{eq} \times Cost_p^3 \\ &a_{eq} = 4339.24 \\ &b_{eq} = 0.054383 \\ &c_{eq} = -4.18 \times 10^{-10} \\ &d_{eq} = 2.69 \times 10^{-18} \end{aligned} \tag{20}$$

$$\begin{aligned} &Cost_{el} = a_{el} + b_{el} \times (h_p \times Q) + c_{el} \times (h_p \times Q)^2 + d_{el} \times (h_p \times Q)^3 \\ &a_{el} = 28278.1 \\ &b_{el} = 286.957 \\ &c_{el} = -0.04365 \\ &d_{el} = 3.2 \times 10^{-6} \end{aligned} \tag{21}$$

$$\begin{aligned} &Cost_d = L \times (a_d + b_d \times D + c_d \times D^2 + d_d \times D^3) \\ &a_d = 7.21125 \\ &b_d = -32.2491 \\ &c_d = 86.3169 \\ &d_d = -48.8413 \end{aligned} \tag{22}$$

The annual operation cost derived from the required energy for pumping the water may be regarded at each pump station as:

$$Cost_e = C_u \times E_p \tag{23}$$

Where:

$$E_p = g_w \times \frac{Q \times h_p}{1000h} \times T \tag{24}$$

In which, $C_u$ is the unit cost of electricity; $E_p\left(KW - hr\right)$ is the annual electricity consumption; $g_w\left(N/m^3\right)$ is the specific weight of water; $Q\left(m^3/s\right)$ is the pumped water flow per hour; $h_p\left(m\right)$ is the pumping head; $h$ is the pumping efficiency; and $T$ is the total hours of pumping in a year.

To calculate the total cost of the assumed system, all explained costs at any node or reach may be incorporated in a unit expression as:

$$Total\,Cost = CRF \times \left( \sum_{n=1}^{NN} Cost_p + Cost_s + Cost_{eq} + Cost_{el} + \sum_{i=1}^{NR} Cost_d \right) + \sum_{n=1}^{NN} Cost_e \quad (25)$$

Where, $CRF$ is Capital Recovery Factor and computed as:

$$CRF = \frac{i(1+i)^n}{(1+i)^n - 1} \quad (26)$$

In which, $i$ is the inflation rate; and $n$ is the estimated length of operation period.

Deep attention to above expressions, Eq. 25 may be paraphrased as follow to derive Eq. 10:

$$\begin{aligned} &Total\,Cost = CRF \times \left( \sum_{n=1}^{NN} Cost_p + Cost_s + Cost_{eq} + Cost_{el} + \sum_{i=1}^{NR} Cost_d \right) + \sum_{n=1}^{NN} Cost_e \\ &Total\,Cost = \left\{ CRF \times \left( \sum_{n=1}^{NN} Cost_p + Cost_s + Cost_{eq} + Cost_{el} \right) + \sum_{n=1}^{NN} Cost_e \right\} + \left\{ CRF \times \sum_{i=1}^{NR} Cost_d \right\} \\ &Total\,Cost = C_n(hp_n) + C_i(D_i) \end{aligned} \quad (27)$$