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TOPOLOGY OPTIMIZATION OF STRUCTURES UNDER TRANSIENT LOADS

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ABSTRACT

In this article, an efficient methodology is presented to optimize the topology of structural systems under transient loads. Equivalent static loads concept is used to deal with transient loads and to solve an alternate quasi-static optimization problem. The maximum strain energy of the structure under the transient load during the loading interval is used as objective function. The objective function is calculated in each iteration and then the dynamic optimization problem is replaced by a static optimization problem, which is subsequently solved by a convex linearization approach combining linear and reciprocal approximation functions.

The optimal layout of a deep beam subjected to transient loads is considered as a case study to verify the effectiveness of the presented methodology. Results indicate that the optimal layout is dependant of the loading interval.

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KEY WORDS: topology optimization; structural optimization; equivalent static loads; strain energy; transient loads.

1. INTRODUCTION

Topology optimization is regarded as the most general type of optimization problems [1]. Topology optimization has been a very active area of research and various methods have been proposed to deal with optimal topology problems subjected to static loads [2-6].

Perhaps, the most well known method is Solid Isotropic Material with Penalization

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(SIMP) [7]. In SIMP, the intermediate designs are penalized so that the optimal design consists of just two types of elements: the elements with 0 density and elements with a fixed maximum value of density. An excellent review of SIMP and its applications can be found in the book by Bendsøe and Sigmond [7].

Compared to static problems dealing with the steady state response, the dynamic problems are concerned with the transient response, which is time dependant [8]. In dynamic problems, if the frequency of excitation is less than one-third of the structure's lowest natural frequency of vibration, then the effects of inertia can be neglected and the problem can be treated as quasi static one [9].

In real world, many loads have a dynamic or transient nature excitation with frequencies of higher than the above mentioned criterion. Therefore, many practical problems may require dynamic analysis to predict the structural responses accurately. The number of articles dealing with optimal topology design under transient or dynamic loads are, however, fairly limited and are far less than the articles dealing with static topology optimization problems. A review of the most relevant articles is presented in the next section.

2. LITERATURE REVIEW

Min *et al.* (1999) formulated a dynamic problem and studied effect of frequency of excitation of impulsive-type loadings on optimal topology of structures. The mean dynamic compliance of structure during the loading interval is used as the objective function and the differences of optimal topologies in dynamic and static cases were highlighted [8].

Jang, Lee and park formulated a dynamic optimization problem and showed that for a structure with a low natural frequency (in comparison with forced frequency), the dynamic characteristics and inertia effect should be considered. They used Equivalent static loads (ESLs) method to simplify the problem [10]. ESLs method was originally proposed by Choi and Park [11]. This method generates static loads having the same displacement field as dynamic loads at each time step. They showed that using ESLs method reduces the complexity of dynamic problems and makes it possible to apply a static optimization algorithm for solving the problem. Kang *et al.* used ESLs concept to minimize the maximum stress of a rotating bar [12]

Maybe the most challenging step in formulating an optimization problem is defining the appropriate target or objective function. As Christensen and Klabring [1] mention, distributing material to make stiffest structure, cause a uniform distribution of strain energy. Evolutionary Structural Optimization (ESO), proposed by Xie and Steven [2], is based on this phenomena. In ESO, the elements with lowest strain energy are removed and the eliminating process is continued until the uniformity of strain energy in the structure is achieved. A complete review of this technique can be found in the book by Huang and Xie [5]. Rouhi and Rais-rohani proposed an almost similar technique named Element Exchange Method (EEM) [6], where low-strain energy elements are exchanged with high-strain energy elements and hence a better load path is created. The above mentioned methods have been applied for static optimization problems.

Even for dynamic problems, it makes sense to make the structures as stiff as possible and to use the concept of uniform strain energy field. In most studies in this area, the mean compliance of structures is used as the objective function with the goal of minimizing the mean strain energy during the loading interval. However, in reality, there is a critical moment that the strain energy reaches a maximum value. Hence, in this article, we propose to use the maximum strain energy of the structure during the loading interval as the objective function. An efficient technique is presented to link the main optimization code with a static optimization subroutine. A numerical example of a deep beam subjected to half sine load is used to evaluate the performance of proposed methodology. The effectiveness of the presented methodology is verified by studying convergence plots, the diagrams of strain energy during the optimization process.

3. FORMULATION OF THE PROBLEM

A deep beam subjected to a half sine load as shown in figure 1 is considered. The force ***F(t)*** is a time varying sin function which is applied on the structure from an initial time $\boldsymbol{t_i}$ to a final time $\boldsymbol{t_f}$**.**

If the design domain is divided into smaller elements of densities x_i, the vector $\boldsymbol{X}$, comprising the thicknesses of all elements, can be used as the optimization or design variables. The objective function can be defined in different ways. The most generally used objective function , to be minimized, in problems with static loads is the compliance. Minimizing the compliance results in optimal distribution of a certain available volume of material to obtain the stiffest structure. The compliance is defined as follows:

$$\mathrm{C} = \boldsymbol{F}^T \boldsymbol{u}(\boldsymbol{x}) \tag{1}$$

Where $\boldsymbol{F}$ is the applied external forces and $\boldsymbol{u}$ represents the displacements. The compliance is defined as the external work of applied forces, which is also equalivalent to the strain energy of structure, defined as:

$$U = \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\,\boldsymbol{u} \tag{2}$$

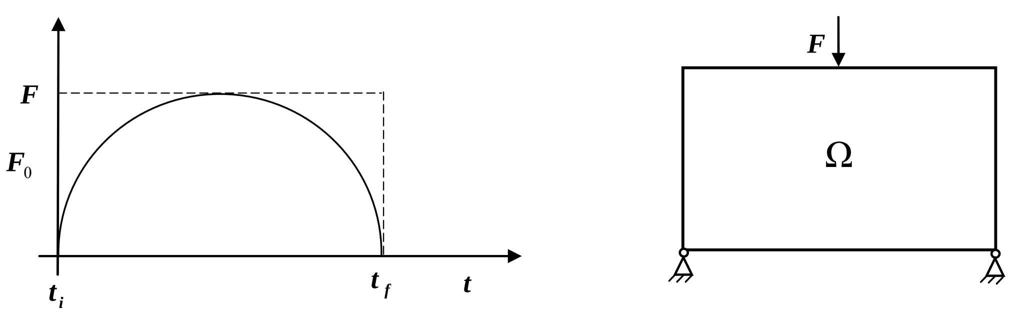

Figure 1. The structure and the applied loading

where U represents the strain energy and $\boldsymbol{K}$ is the stiffness matrix of structure.

In case of dynamic loads, the strain energy is a time varying function with different values at different times and therefore using the compliance as the objective function is more complicated. In this article, the maximum value of strain energy is used as the objective function. Minimizing the maximum strain energy of structure during loading time interval, between $\boldsymbol{t_i}$ and $\boldsymbol{t_f}$, guarantees that at all other time steps, the structure has a smaller strain energy. The optimization problem can therefore be formulated as:

$$\begin{aligned} & \mathit{find}\ X \\ & \mathit{to\,minimize}\ \left(\max U\right) \\ & \mathit{subject\,to} \begin{cases} \sum_{i=1}^{n} x_i a_i \le V \\ 0 < x_i^{min} \le x_i \le x_i^{max} \end{cases} \end{aligned} \tag{3}$$

Where V is volume of available material, a_i is area of element i, x_i represents the thickness of element i with corresponding lower and upper bounds of x_i^{min} and x_i^{max} respectively and n is number of elements.

One of the difficulties arising in dynamic analysis is the effect of added inertia force on the structure's responses. The equivalent static loads (ESLs) can be used to simplify the analysis [13]. ESLs are defined as the static loads generating the same response fields as those under a dynamic load at an arbitrary time of dynamic analysis [13]. The equivalent static loads method has been previously used for linear dynamic response optimization [14-16]. Using the ESLs provides the advantage that the problem can be defined as a topology optimization problem subjected to static loads and therefore any of existing optimization algorithms, suitable for static problems, may be used.

To use ESLs, the differential equation of motion must first be solved to obtain the displacement fields. The governing differential equation of an undamped forced vibration problem is expressed as [17]:

$$\boldsymbol{m}(X)\boldsymbol{u}(t) + \boldsymbol{K}(X)\boldsymbol{u}(t) = \boldsymbol{F}(t) \tag{4}$$

Where $\boldsymbol{m}$ is the mass matrix, $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{F}$ is the vector of applied forces, and $\boldsymbol{u}$ and $\ddot{\boldsymbol{u}}$ are displacement and acceleration vectors respectively. To obtain the equivalent static loads, $\boldsymbol{F}_{eq}$, at time t=s we can write:

$$\boldsymbol{F_{eq}}(s) = \boldsymbol{Ku}(s) \tag{5}$$

It should be noted that even if the external force is applied to a single point of a structure, the equivalent static loads are applied to all degrees of freedom of the structure [10]. As the stiffness matrix is known, it is only necessary to calculate the displacements by solving the governing differential equation.

4. SOLUTION OF GOVERNING EQUATION

A time history analysis provides the response of a structure over time during and after the application of a load. In order to find the full time history response of a structure, the equation of motion must be first solved.

The central finite difference formula was used to estimate the displacement and acceleration at time t_n. The central finite difference formula approximates velocity and acceleration by using Taylor series expansions of u_{n+1} and u_{n-1} at time t_n as:

$$\dot{u}_n = \frac{1}{2\Delta t}(u_{n+1} - u_{n-1}) \tag{6}$$

$$\ddot{u}_n = \frac{1}{\Delta t^2}(u_{n+1} - 2u_n + u_{n-1}) \tag{7}$$

Substituting these expressions in the governing differential equation, we then obtain:

$$(\frac{1}{\Delta t^2}m)u_{n+1} = f_n - (k - \frac{2}{\Delta t^2}m)u_n - (\frac{1}{\Delta t^2}m)u_{n-1} \tag{8}$$

Different steps for solving the governing equation using central difference scheme can be summarized as follows: 1) calculation of mass and stiffness matrices, 2) initializing the displacement u_0 and acceleration $\ddot{u}_0$, 3) selecting the time step Δt, 4) calculation of $u_{-1} = u_0 - \Delta t u_0 + \Delta t^2 \ddot{u}_0/2$, 5) using equation (8) to calculate u_{n+1}, u_{n+1}, corresponding displacement at time u_{n+1}

5. CALCULATION OF MASS MATRIX

A mass matrix is a discrete representation of a continuous distribution of mass [9]. The simplest method of calculation is using the particle masses. The process is called mass lumping and results in a diagonal mass matrix. For Q_4 elements used in this article, the lumped mass matrix can be written as:

$$\boldsymbol{m_l} = \frac{m}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

where m is the mass of one element. The so-called consistent mass matrix is another approach which can be used for calculation of mass matrix:

$$\boldsymbol{m}_c = \int \rho \boldsymbol{N}^T \boldsymbol{N}\, dv \tag{10}$$

where ρ is the mass density of material and $\boldsymbol{N}$ is the shape function. This mass matrix is called consistent because it uses the same shape function used in developing the stiffness matrix. For the Q_4 elements, the consistent mass matrix is expressed as:

$$\boldsymbol{m}_c = \frac{m}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \tag{11}$$

Cook (2001) recommends to use a weighted average of the lumped and consistent mass matrix as follows [8].

$$\boldsymbol{m} = (1-\beta)\boldsymbol{m}_l + \beta \boldsymbol{m}_c, \qquad \beta = 0.5 \tag{12}$$

We have used this definition in calculating the element mass matrices. The mass matrix of the structure is then obtained by assembling the element mass matrices.

6. MAIN OPTIMIZATION ALGORITHM

The detailed steps of the main optimization algorithm are shown in Figure 2.

The main steps of the algorithm can be described as: 1) dynamic analysis, 2) calculating ESLs, and 3) topology optimization using static loads. In the first step, the vector of nodal displacements at all time steps are calculated. In each time step, the corresponding ESLs are then calculated. At a critical time step (t*), when the structure has the maximum strain energy, the corresponding critical ESLs, denoted by $F_{eq}^*(t^*)$, are then calculated as follows:

$$\left(F_{eq}^*(t^*)\right)^T \boldsymbol{u}^*(t^*) = Max\ \left\{ \boldsymbol{F}_{eq}(t)\boldsymbol{u}(t) \qquad for\ \ t = t_i,\ldots t_f \right\} \tag{13}$$

Finally, the structure is optimized using the critical ESLs and updated element thicknesses are determined. This process is repeated until the convergence is achieved. It is important to note that in our proposed technique, the last step can be done by using any existing optimization subroutine written for static loads. The static optimization algorithm used in this article is described in the following.

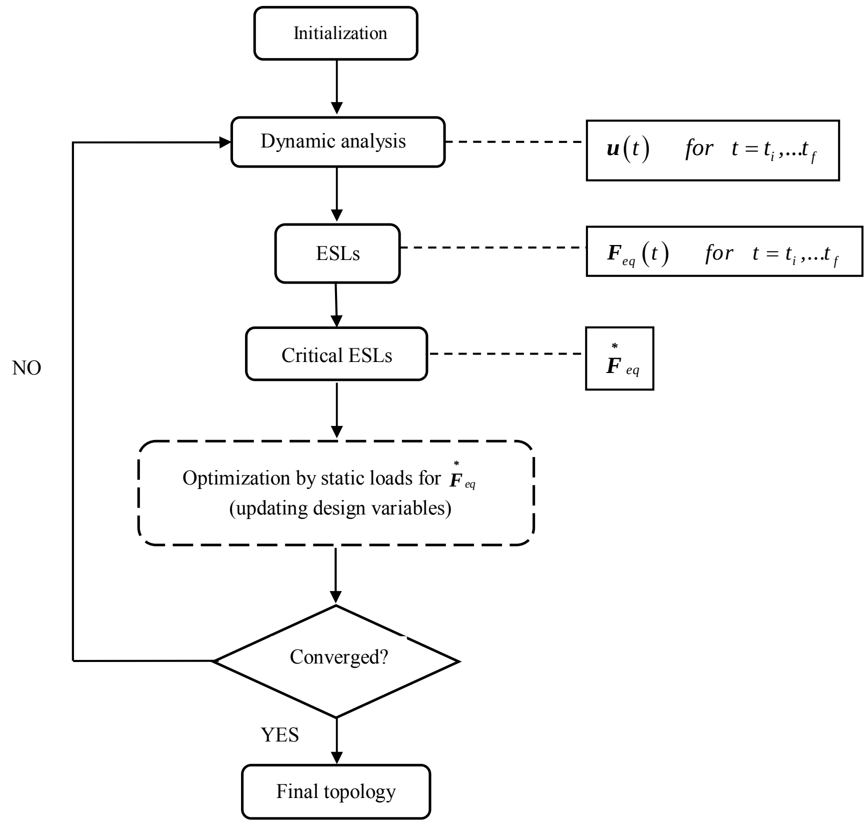

Figure 2. The flowchart of main optimization algorithm

7. OPTIMIZATION ALGORITHM FOR STATIC LOADS

As mentioned above in each iteration of main algorithm, an optimization subroutine is run, This step is distinguished in figure 2 by a dotted rectangle. The optimization problem can be formulated as follows:

$$\begin{aligned} & find\ X \\ & to\min imize\ \left(\overset{*}{U}\right) \\ & subject\ to \begin{cases} K(X)u = F \\ \sum_{i=1}^{n} x_i a_i \le V \\ 0 < x_i^{min} \le x_i \le x_i^{max} \end{cases} \end{aligned} \tag{14}$$

Where $\overset{*}{U}$ is the strain energy caused by F_{eq}^{*} , calculated from equation (13). Therefore

$\overset{*}{U}$ can be determined as follows:

$$\overset{*}{U} = F_{eq}^{*T} u^{*} \tag{15}$$

Other parameters are the same as before.

In this step, F_{eq}^{*} is fixed and u^{*} depends only on the thicknesses of elements and is not a function of time. Therefore, any of common optimization algorithms, appropriate for solving static problems, may be applied here. In this research, the convex linearization approach (CONLIN) combining linear and reciprocal approximation functions [1] is used. CONLIN is a sequential approximation algorithm that starts with an initial variable X^0. In each iteration k, the objective function and all constrains are linearized at the design point X^k in terms of either x_j or $1/x_j$, $j = 1,...,n$, where n is the number of elements. The linear approximation is performed in terms of x_j if the corresponding gradient is positive. Otherwise, the linearization is performed in terms of reciprocal variable $1/x_j$. The gradient of objective function at x_j is calculated from the following equation:

$$\frac{\partial U^{*}}{\partial x_j} = -u(x)^T K_j^0 u(x) \tag{16}$$

Where K_j^0 is the global version of element j stiffness matrix, per unit thickness. As can be noted from the equation 16, the components of the gradient are not positive and therefore the objective function in iteration k is linearized in terms of reciprocal variables. This results in [1]:

$$U^{*k}\left(\boldsymbol{X}\right) = U^{*}\left(\boldsymbol{X}^k\right) + \sum_{j=1}^{n} \frac{\partial U^{*}\left(\boldsymbol{X}^k\right)}{\partial x_j} \frac{x_j^k\left(x_j - x_j^k\right)}{x_j} \tag{17}$$

Lagrangian duality can be used for solving this subproblem [e.g. 18]. This process is repeated until a defined convergence criterion is satisfied. We have defined the convergence criterion as the maximum value of changes in the values of design variables (i.e. thicknesses of elements) from the previous iteration:

$$\max\left\{| x_j^{k+1} - x_j^{k} |,\ j = 1,...,n\right\} \le \Delta \tag{18}$$

8. NUMERICAL EXAMPLE

The proposed methodology is applied to a two-dimensional plate problem. The rectangular area of 100x50 cm , shown in Figure 1, is used as the design domain. The design domain is discretized using 1250 Q_4 elements of 2x2 cm squares and a half Sinus load pattern was applied at the upper middle of the plate. The material properties of the isotropic material are as follows: Young modulus $E=2.1\text{x}10^6$ N/mm^2 and Poisson's ration ν=0.3.

The objective of the optimization problem is to find the material distribution that minimizes the maximum strain energy of structure during all time intervals. The design constraints are the maximum volume of used materials and the bounds for density of each element. The optimization problem can then be formulated as equation 14, in which the minimum and maximum thicknesses of elements are 0.001 and 5 respectively. In order to investigate the effect of loading interval of the half sin load on the final optimal layout, six examples were analyzed. In these examples, we have varied the loading interval of applied half sine load by changing the value of t_f (Figure 1). It should be noted that a large value of t_f corresponds to uniform load and a small t_f resembles an impulse load. The six values of t_f are chosen by trial and error to show the important cases of various possible topologies.

The optimal topologies corresponding to six values of t_f are shown in Cases A to F of Figure 3 respectively. The red and blue colors show the presence and absence of the material.

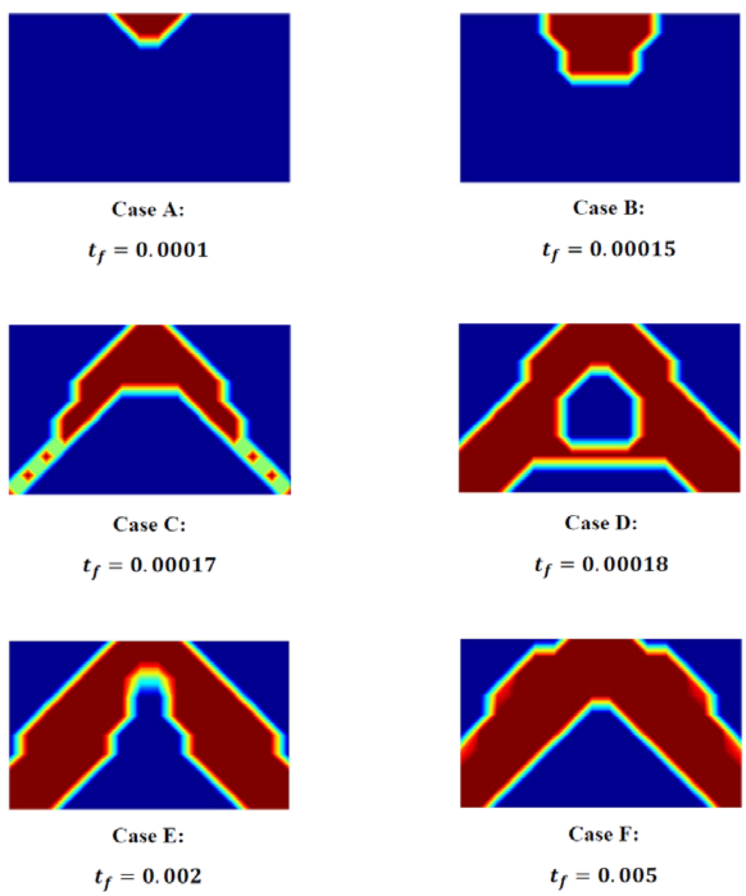

Figure 3. Topology optimization of the example for different values of t_f.

It can be noted that as the t_f increases, the optimal topology layout becomes similar to the one for static loading, which implies the transition of the problem from the transient response to the steady state response. The optimal layout topology of case F (Figure 3) is very similar to optimal layout under static loading reported in [9].

As mentioned before our goal is to minimize the maximum strain energy of the structure. In order to check the accuracy of the developed program, the study of convergence plots, showing the variation of the objective function during the optimization process, is useful. In Figure 4, these plots are presented for cases A to F.

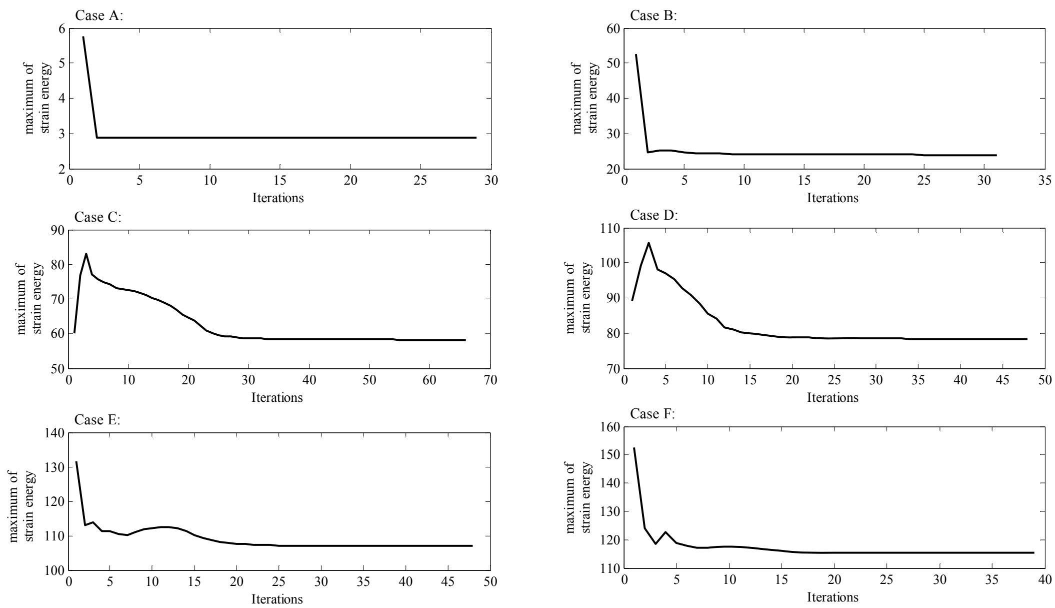

Figure 4. Convergence history for cases A to F.

The convergence history plots (Figure 4) indicate that in all six cases, the maximum of strain energy is decreasing during the optimization process and is converged to minimum values. The minimization of maximum value of strain energy guarantees that the values of strain energy at all other times are larger than the final value of converged objective function.

The optimal design variables result in the stiffest structure and, in turn, in a uniform pattern of strain energy in the structure. To better understand advantage of obtaining a uniform strain energy pattern, the case of a one dimensional structure is discussed in the following. In a 1-D structure, uniformity of strain energy is equivalent to uniformity of stress. Therefore, when the final topology of such structure is subjected to a transient load, the maximum stress, which occurs in a critical moment, is uniformly distributed in all parts of structure. Therefore, if such a structure subjected to the failure load, all elements reach the ultimate stress simultaneously and therefore the best use of materials is achieved.

9. SUMMARY AND CONCLUSIONS

In this article, optimal topology layout of structures subjected to transient loads are obtained. In case of dynamic loading, we have a time varying response and using the compliance as the objective function is more complicated than is the case for static loads. As the strain energy of structure is also varying with time, its maximum value at an arbitrary time during the loading interval is used as the objective function. This choice guarantees that at all other times, the structure has a larger value of strain energy. The concept of ESLs is used to reduce the required computational effort.

The proposed methodology was verified on a deep beam subjected to a transient half Sin load for varying loading intervals. Results of the analysis for six different cases demonstrate that the optimal layout is very much depends on the loading interval and that the maximum value of strain energy is decreasing with time.

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