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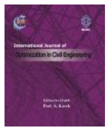

NUMBER OF SPANNING TREES FOR DIFFERENT PRODUCT GRAPHS

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ABSTRACT

In this paper simple formulae are derived for calculating the number of spanning trees of different product graphs. The products considered in here consists of Cartesian, strong Cartesian, direct, Lexicographic and double graph. For this purpose, the Laplacian matrices of these product graphs are used. Form some of these products simple formulae are derived and whenever direct formulation was not possible, first their Laplacian matrices are transformed into single block diagonal forms and then using the concept of determinant, the calculations are performed.

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1. INTRODUCTION

Many practical large-scale structures and in particular space structures are either regular or can be made regular by addition or reduction of some member and/or nodes [1]. For optimal static and dynamic analyses of such structures some matrices are involved for which the eigenvalues and eigenvectors should be calculated, through which the inverse of the corresponding matrices can easil;y be obtained. The models of regular structures are often in the form of the product of two graphs. For this reason one can establish a meaningful

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relation between the structural matrices and the corresponding graph matrices. The graph product can be utilized in recognizing the generators of these product graphs. For this problem different product graphs are defined. There are also efficient methods for the evaluation of eigenvalues and the inversion of the graph matries such as the adjacency and Laplacian matrices. The eigenvalues of the Laplacian matrices can be used in bisection or nodal ordering of the graphs for profile reduction. The use of Fiedler vector is an example of such applications. This matrix has also many applications in the theory of graph itself. One such application belongs to the calculation of the number of spanning tree of the graphs. This number can be utilized in nodal ordering of graphs for optimal analysis. Here we use the relationship develoed for eigensolution of regular structures in order to find the number of spanning trees by means of closed form solutions.

The number of spanning trees of a graph G is known as the *complexity* of the graph, and it is denoted by $\tau(G)$. This number can be found using the determinant of the Laplacian matrix of a graph from which one arbitrary row and column is deleted [2]. Two different algebraic methods re available for calculating the complexity of graphs [3, 4]. Considering the Laplacian matrix L of a graph, and using its determinant, the number of spanning trees can be found by the following relationship [3]:

$$\tau(K_n) = \frac{1}{n^2} det(L + J) \tag{1}$$

Using the eigenvalues, the following relationship is presented in [4] for calculation of the complexity of G:

$$\tau(G) = \frac{1}{n}\prod_{j=2}^{n} \mu_j \tag{2}$$

where, n is the number of nodes, and J is a matrix having dimensions identical to L with all the entries being equal to unity. In Eq. (2) μ_j are the eigenvalues of the Laplacian matrix L, and obviously $\mu_1 = 0$ should not be included. Here, we use mostly Eq. (2) for evaluating the complexity τ(G), though in one section we obtain this relationship in terms of the determinants. In general, when ever the calculation of the eigenvalues is simpler, Eq. (2) is directly employed. However, since we need the products of the eignevalues, Eq. (2) will be expressed in terms of the determinant of a block matrix. Using Eq. (1) is simpler when the eigenvalues or the determinant of L+J can easily be calculated. As an example, if G is a complete graph, then obviously L+J = nI with I being the unit matrix. Therefore, the determinant is the products of the diagonal entries and we have

$$\tau(K_n) = \frac{1}{n^2} det(nI) = \frac{n^n}{n^2} = n^{n-2} \tag{3}$$

This relationship is the same as the Caley's relationship which is proved by Prufer [5], using the concept of edge contraction. For some special products some formulae are derived [6], and here we present these in a complete form for different graph products.

2. DEFINITIONS OF DIFFERENT GRAPH PRODUCTS

2.1 Cartesian Product of Two Graphs

Many models in engineering have regular patterns and can be viewed as the Cartesian product of a number of simple graphs. These subgraphs, which are used in the formation of a model, are called the *generators* of that model.

The simplest Boolean operation on a graph is the Cartesian product $K \times H$ introduced by Sabidussi [7]. The Cartesian product is a Boolean operation $G = K \times H$ in which for any two nodes $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $N(K) \times N(H)$, the member uv is in $M(G)$ whenever,

$$\begin{gathered} u_1 = v_1 \text{ and } u_2v_2 \in M(H), \text{ or} \\ u_2 = v_2 \text{ and } u_1v_1 \in M(K) \end{gathered} \tag{4}$$

2.2 Strong Cartesian Product of Two Graphs

This is another Boolean operation, known as the *strong Cartesian product*. The strong Cartesian product is a Boolean operation $G = K \boxtimes H$ in which, for any two nodes $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $N(K) \times N(H)$, the member uv is in $M(G)$ if:

$$\begin{gathered} u_1 = v_1 \text{ and } u_2v_2 \in M(H), \text{ or} \\ u_2 = v_2 \text{ and } u_1v_1 \in M(K), \text{ or} \\ u_1v_1 \in M(K) \text{ and } u_2v_2 \in M(H) \end{gathered} \tag{5}$$

2.3 Direct Product of Two Graphs

This is another Boolean operation known as the *direct product* introduced by Weichsel [8], who referred to it as the *Kronecker Product*. The direct product is a Boolean operation $G = K*H$ in which for any two nodes $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $N(K) \times N(H)$, the member uv is in $M(G)$ if:

$$u_1v_1 \in M(K) \text{ and } u_2v_2 \in M(H) \tag{6}$$

2.4 Lexicographic Product of Two Graphs

This is another Boolean operation known as the *lexicographic product* introduced by Harary [9], and occasionally it is referred to as the *composition product*. The lexicographic product is a Boolean operation $G = K \circ H$ in which for any two nodes $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $N(K) \times N(H)$, the edge uv is in $M(G)$ if:

$$\begin{gathered} u_1v_1 \in M(K) \text{ or} \\ u_1 = v_1 \text{ and } u_2v_2 \in \mathrm{M(H)}. \end{gathered} \tag{7}$$

More concretely, the lexicographic product can be formed by replacing each node of *K* with a copy of *H* and drawing all possible edges between adjacent copies.

2.5 Double of graph

The double of a simple graph *G* is defined as

$$D(G) = G \times T_2 \tag{8}$$

where T_2 is a complete graph K_2 with one loop added to each of its nodes [10].

3. NUMBER OF SPANNING TREES USING EIGENVALUES

First we consider the Cartesian product. For this product we have

$$M_{mn} = F_n(A_m, B_m, C_m) = I_n \otimes (A+B)_m + T_n \otimes (-B_m) \tag{9}$$

where

$$F_n(A_m, B_m, C_m) = \begin{bmatrix} A_m & B_m & & & & \\ B_m & C_m & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & C_m & B_m \\ & & & & B_m & A_m \end{bmatrix}_n \tag{10}$$

and $T_n = F_n(1,-1,2)$

Since $IT = TI$, therefore diagonalization is possible and we have

$$eig(M_{mn}) = \bigcup_{i=1}^{n} \{ eig[(A+B)_m - \lambda_i(T_n) B_m] \} \tag{11}$$

Here for the product $P_m \times P_n$ we have

$$(A+B)_m = F_n(1,-1,2) = T_m \quad ; \quad B_m = -I_m \tag{12}$$

Therefore it is sufficient to add the eigenvalues of P_m to those of P_n

$$\begin{aligned}
&\lambda_m^P = 2 + 2\cos\frac{k\pi}{m} = 2\cos^2(\frac{k\pi}{2m}) \;\; ; \;\; k = 1:m \\
&\lambda_m^P + \lambda_n^P = 2(\cos^2(\frac{k\pi}{2m}) + \cos^2(\frac{t\pi}{2n})) \;\; ; \; k = 1:m\,; \; t = 1:n
\end{aligned} \tag{13}$$

According to Eq. (2), these values should be multiplied using different values of *k* and *l*, however, *k* and *l* should not be maximal simultaneously otherwise in that case the eigenvalues will be zero and should be discarded. Thus

$$\tau(P_m \times P_n) = \frac{4^{mn-1}}{mn}\prod_{k=1}^{m}\prod_{t=1}^{n}(\cos^2(\frac{k\pi}{2m}) + \cos^2(\frac{t\pi}{2n})) \tag{14}$$

The number of multiplications will be *mn* and one of them should be discarded. Thus the power of 4 will be (mn–1).

For the product $P_m \times C_n$ similar operations should be performed, with the only difference that

$$\lambda_n^C = 2 - 2\cos\frac{2t\pi}{n} = 2\sin^2(\frac{t\pi}{n}) \;\; ; \;\; t = 1:n \tag{15}$$

In this way it is sufficient to change *Cos* to *Sin* in Eq. (14) in the second term which corresponds to C_n. In this equation the divisor 2*n* should also be replaced by *n*. Then we will have

$$\tau(C_n \times P_m) = \frac{4^{mn-1}}{mn}\prod_{k=1}^{m}\prod_{t=1}^{n}(\cos^2(\frac{k\pi}{2m}) + \sin^2(\frac{t\pi}{n})) \tag{16}$$

By similar operations mentioned above, for $C_n \times C_m$ we will have

$$\tau(C_n \times C_m) = \frac{4^{mn-1}}{mn}\prod_{k=1}^{m}\prod_{t=1}^{n}(\sin^2(\frac{k\pi}{m}) + \sin^2(\frac{t\pi}{n})) \tag{17}$$

Here we use a similar method for finding the number of spanning trees of a lexicographic product, since one can easily find the corresponding eigenvalues.

According to [11] the eigenvalues for the Laplacian matrix of G_noH_m can be calculated using the following simple algorithm:

1. Calculate the eignvalues for the Laplacian matrices of G_n and H_m and delete the zero eigenvalue from those of H_m.

2. Multiply the vector $d(G_n)$, which represents the degree of the nodes of G_n, by m and find the pair wise Cartesian sum of the result by the eigenvalues of H_m found the previous step. Call the results as the group 1 results.

3. Multiply the eigenvalues of G_n found in Step 1 by *m*, and obtain the group 2 results.

Now suppose the product is in the form of C_noP_m. In this case

$$\lambda_m^P = 2 + 2\cos\frac{k\pi}{m} \;\; ; \;\; \lambda_n^C = 2 - 2\cos\frac{2t\pi}{n}, \;\; d(C_n) = \{2,2,...,2\}_n \tag{18}$$

In this case the group 1 results will be as

$$m\{2,2,...,2\}_n \oplus \{2 + 2\cos\frac{k\pi}{m}\} \tag{19}$$

For the group 2 results we have

$$m\{2 - 2\cos\frac{2t\pi}{n}\} \tag{20}$$

In the group 1 we will have n terms as $2m + 2 + 2\cos\frac{k\pi}{m}$, and using Eq. (2) we obtain

$$\tau(G) = \frac{1}{n}\prod_{j=2}^{n}\mu_j = \frac{1}{mn}\left[2^{n(m-1)}\prod_{k=1}^{m-1}(m + 1 + \cos\frac{k\pi}{m})^n \times (4m)^{n-1}\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{n}\right] \tag{21}$$

The power m–1 is present because of deleting zero in the first step of the calculation, and the power n–1 is because of deleting $\mu_1 = 0$ and μ_1 corresponds to the last number of the group 2 results, which is not included.

Now we investigate some special cases:

If the graph is a complete graph G_n, then the group 1 and group 2 results will be as follows:

$$1 \Rightarrow m\{n-1, n-1, ..., n-1\}_n \oplus \{2 + 2\cos\frac{k\pi}{m}\} \quad ; \quad 2 \Rightarrow m\{0, n, n, ..., n\}_n \tag{22}$$

If all these terms are pair wise multiplied, after deleting $\mu_1 = 0$ and dividing the result by the total number of nodes, we will have

$$\tau(K_n \circ P_m) = (mn)^{n-2}\prod_{k=1}^{m-1}(mn - m + 2 + 2\cos\frac{k\pi}{m})^n \tag{23}$$

One can rewrite a complete graph as $K_n = K_n$oP_1, therefore in Eq. (23) choosing m=1 we obtain, $\tau(K_n) = n^{n-2}$, which is applicable to complete graphs.

Obviously similar relationships can be obtained for G=C_noC_m.

Now we employ Eq. (25) for the strong Cartesian product, and as an example we study $G = C_m \boxtimes C_n$. For this case we have

$$M_{mn} = F_n(A_m, B_m, C_m);\ A_m = C_m = G_m(8,-1,8);\ B_m = G_m(-1,-1,-1) \tag{24}$$

where

$$G_n(A_m, B_m, C_m) = \begin{bmatrix} A_m & B_m & & & & B_m \\ B_m & C_m & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & C_m & B_m \\ B_m & & & & B_m & A_m \end{bmatrix}_n \tag{25}$$

and

$$A_m = 9I_m + B_m; M_{mn} = G_n(9I_m + B_m, B_m, 9I_m + B_m) = 9I_n \otimes I_m + G_n(1,1,1) \otimes B_m \tag{26}$$

Therefore

$$\begin{aligned} eig\ (M_{\mathrm{mn}}) &= \bigcup_{k=1}^{m}[eig\{9\mathbf{I}_n - (1+2\cos\frac{2k\pi}{m})\mathbf{G}_n(1,1,1)\}] \\ &= 9 - (1+2\cos\frac{2k\pi}{m})(1+2\cos\frac{2t\pi}{n}) \quad ; k = 1:m; t = 1:n \end{aligned} \tag{27}$$

The reason for this decomposition is that

$$A_i A_j = A_j A_i \quad ; \quad B_i B_j = B_j B_i \tag{28}$$

Therefore

$$\tau(C_m \otimes C_n) = \frac{1}{mn}\prod_{k=1}^{m}\prod_{t=1}^{n}\{9 - (1 + 2\cos\frac{2k\pi}{m})(1 + 2\cos\frac{2t\pi}{n})\} \tag{29}$$

In the above product we should not have *k* and *t* as maximum simultaneously.

Here we consider another product known as the double graph [10]. First the method for calculating the eigenvalue of their Laplacian is presented and then the number of their spanning trees is obtained.

A double graph can be considered as $D[G] = G \circ N_2$. This means that a double graph is special case of the lexicographic product, where N_m represents m isolated nodes. For this case we can write

$$M_{mn} = I_m \otimes 2D_n - O_m \otimes A_n \tag{30}$$

where D is a diagonal matrix and the entries on the diagonal are the degrees of the graph, and O is a matrix with all its entries being 1. For this case, we have $\mathrm{I} \times \mathrm{O} = \mathrm{O} \times \mathrm{I}$ and one can use decomposition for finding the eigenvalue.

Instead of this, one can use the same relationship which we derived for the lexicography product. It should be noted that here the eigenvalues of H_m contains m zeros and we should delete all of these in the first step of calculations. Since here we have m = 2, therefore the union of the eigenvalues of group 1 and group 2 will be as follows:

$$eig\,(D(G_n)) = 2(eig(G_n) \cup d(G_n)) \tag{31}$$

This means that we should put the eigenvalues and the degrees of the nodes next to each other and double them. Thus if we delete the zero eigenvalue from the eigenvalues of G_n and we multiply by 2, then dividing the result to the number of nodes one can obtain the number of spanning trees.

Here the number of nods was $2n$, and if the degrees of G_n be denoted by $d_1, d_2, \ldots, d_n$, then the following simplified relationship will be obtained:

$$\tau\,(D(G_n)) = \frac{1}{2n}2^{n-1}\tau(G_n)2^n nd_1d_2d_3...d_n = 2^{2n-2}d_1d_2d_3...d_n\tau(G_n) \tag{32}$$

4. SOME SPECIAL CASES

1. If G is a compete graph with k nodes, then we have

$$\tau\,(D(G_n)) = 2^{2n-2}k^n\tau(G_n) \tag{33}$$

2. If G is path graph, then obviously $\tau\,(P) = 1$. This means that we have only one spanning tree. In this case, since the degrees of the ends are 1 and the remaining nodes have a degree of 2, therefore we have

$$\tau\,(D(P_n)) = 2^{2n-2} \times 2^{n-2} \times 1 = 2^{3n-4} \tag{34}$$

3. If G is a cycle, then obviously $\tau\,(C) = n$. This means that in a cycle with n nods, we have n spanning trees. In this case since the nodes are all of degree 2, therefore we have

$$\tau\,(D(C_n)) = 2^{2n-2} \times 2^n \times n = n2^{3n-2} \tag{35}$$

5. THE NUMBER OF SPANNING TREES USING THE DETERMINANT

Here we perform the calculations in a different manner. First we should block diagonalized the Laplacian. The determinant of such a matrix is equal to the product of determinants of its blocks.

Suppose M can be expressed as the sum of two Kronecker products as

$$M = A_1 \otimes B_1 + A_2 \otimes B_2 \tag{36}$$

Now let P be the matrix which diagonalized A_1 and A_2. These matrices are normal matrices.

The necessary and sufficient condition for simultaneous diagonalization of two Hermitian matrices A_1 and A_2 by an orthogonal matrix is that $A_1 A_2 = A_2 A_1$ holds. In this case one can show that $U = P \otimes I$ block diagonalizes the matrix M and w will have [12]:

$$U^t M U = D_{A_1} \otimes B_1 + D_{A_2} \otimes B_2 \tag{37}$$

Therefore for the evaluation of the eigenvalues of M, one can find the eigenvalues of the blocks on the diagonal of this matrix, because if U is orthogonal, then M and U^tMU are two similar matrices. In this way for calculating the determinant of M one can calculate the determinant off a similar matrix, namely U. It is obvious that since U is block diagonal, it is only necessary to calculate the determinants of the blocks and then multiply their magnitudes.

Now suppose the matrix M is the sum of three Kronecker products as

$$M = \sum_{i=1}^{3} A_i \otimes B_j \tag{38}$$

For block diangonalization of such a matrix, similar to the case where we had the sum of two Kroneker products, the commutativity with respect to product for each pair if A_i should hold, i.e.

$$A_i A_j = A_j A_i \quad i, j = 1:3,\ i \neq j \tag{39}$$

For the remaining calculation we need to present some simple relationships for the matrices (blocks) produced on the diagonal. First we consider the following form:

$$H_n = \begin{bmatrix} a & b & & & & \\ c & a & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & a & b \\ & & & & c & a \end{bmatrix}_n \tag{40}$$

Using Ref. [13] we will have

$$det(H_n) = \begin{cases} a^n & bc = 0 \\ (n+1)(a/2)^n & a^2 = 4bc \\ (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) & a^2 \neq 4bc \end{cases} \tag{41}$$

where α and β are the roots of the equation $x^2 - ax + bc = 0$.

Now we obtain the determinants of F and G in terms of the determinant of H_m.

5.1 Determinants of the matrices in the form of F

This matrix has the following form:

$$F_n(a,b,c) = \begin{bmatrix} a & b & & & & \\ b & c & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & c & b \\ & & & & b & a \end{bmatrix}_n \tag{42}$$

For calculating the determinant of this matrix, it can be seen that if ignore the first row and first column, the submatrix left will have form H_m. Now expanding with respect to the first row we will have

$$\det(F_n(a,b,c)) = a\det(\begin{bmatrix} c & b & & & & \\ b & c & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & c & b \\ & & & & b & a \end{bmatrix}_{n-1}) - b^2\det(\begin{bmatrix} c & b & & & & \\ b & c & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & c & b \\ & & & & b & a \end{bmatrix}_{n-2}) \tag{43}$$

Expanding these two matrices with respect to the last row, we obtain H_n form.

$$det(F_n(a,b,c)) = a^2 det(H_{n-2}) - 2ab^2 det(H_{n-3}) + b^4 det(H_{n-4}) \tag{44}$$

The determinant of H_n is as given in Eq. (41).

5.2 Determinants in the form of G

This matrix has the following form

$$G_n(a,b,c)=\begin{bmatrix} a & b & & & & b \\ b & c & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & c & b \\ b & & & & b & a \end{bmatrix}_n \tag{45}$$

Expanding the determinant of this matrix with respect to the first row we will have

$$\det(G_n(a,b,c))=a\det(\begin{bmatrix} c & b & & & & \\ b & c & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & c & b \\ & & & & b & a \end{bmatrix}_{n-1})-b\det(\begin{bmatrix} b & b & & & & \\ & c & b & & & \\ & b & c & . & & \\ & & . & . & . & \\ & & & . & c & b \\ b & & & & b & a \end{bmatrix}_{n-1}) \\ +(-1)^{n+1}b\det(\begin{bmatrix} b & c & b & & & \\ & b & c & b & & \\ & & b & . & & \\ & & . & . & . & \\ & & & . & b & c \\ b & & & & & b \end{bmatrix}_{n-1}) \tag{46}$$

We expand each of the above matrices until we reach to an H_n form.

$$\det(G_n(a,b,c))=a[a\det(H_{n-2})-b^2\det(H_{n-3})]-b[b(a\det(H_{n-3})- \\ b^2\det(H_{n-3}))-(-b)^{n-1}]+(-1)^{n+1}b(b^{n-1}+(-1)^n b\det(H_{n-2}) \tag{47}$$

Ultimately after simplifying the result we obtain the following relationship:

$$\det(G_n(a,b,c))=(a^2-b^2)\det(H_{n-2})-2ab^2\det(H_{n-3})+b^4\det(H_{n-4})-2(-b)^n \tag{48}$$

The determinant of is given in Eq. (41). This means the difference between the determinants of the form F and G is equal to $-b^2\, det(H_{n-2})-2(-b)^n$.

Consider the following block diagonal matrix, where all the submatrices A_i are of dimension m.

$$D_n(A_i)=\begin{bmatrix} A_1 & & & & & & & & \\ & A_2 & & & & & & & \\ & & A_3 & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & A_{n-2} & & \\ & & & & & & & A_{n-1} & \\ & & & & & & & & A_n \end{bmatrix} \tag{49}$$

For calculating the number of spanning trees we have to multiply all the non-zero eigenvalues. Obviously only one of A_is will have zero eigenvalue, since the rank of the Laplacian is only one less than its dimension. We assume this submatrix is A_p. First we calculate the eigenvalues of A_p and we delete its zero eigenvalue. Then the product of the remaining eigenvalues is multiplied in the determinants of the remaining A_is. Thus for any product of two graphs C_n and D_m, where the product is designated by $\bullet$, we will have

$$\tau(C_n \bullet D_m)=\frac{1}{mn}\prod_{i=2}^{mn}\mu_i=\frac{1}{mn}\Big(\prod_{\substack{p=1\\ \lambda_p\neq 0}}^{m}\lambda_p\Big)\Big(\prod_{\substack{i=1\\ i\neq p}}^{n}\det(A_i)\Big);\ \lambda_p=eig(A_p);\det(A_p)=0 \tag{50}$$

Now we investigate different graph products.

For the product $C_n \times P_m$ the Laplacian matrix has the following form:

$$M_{mn}=F_m(A_n,B_n,A_n-B_n)=I_m\otimes G_n(2,-1,2)+F_m(1,-1,2)\otimes I_n \tag{51}$$

Considering Eq. (36) and Ref. [12] one can write:

$$\lambda_M=\bigcup_{i=1}^{n}eig(M_i);M_i=\lambda_i(A_1)B_1+\lambda_i(A_2)B_2 \tag{52}$$

Considering the definition of Eq. (49), and the previous equation, the blocks which will be produced on the diagonal will be as follows:

$$U^tM_{mn}U=D_m(G_n(2,-1,2)+\lambda_iI_n)\quad;\quad\lambda_i=eig(F_m(1,-1,2)) \tag{53}$$

Therefore the matrix A_p will be as

$$A_p=G_n(2,-1,2)\ ;\ \lambda_p=\lambda_i(A_p)=2-2\cos\frac{2i\pi}{n}\ ;\ i=1:n\ ;\ \prod_{\substack{p=1\\ \lambda_p\neq 0}}^{n}\lambda_p=n^2 \tag{54}$$

Considering Eq. (40), we ultimately obtain

$$\tau(C_n \times P_m) = \frac{n}{m}\prod_{i=2}^{m} det(G_n(2,-1,2) + (2+2\cos\frac{i\pi}{m})I_n) \qquad (55)$$

where the determinant of this matrix is given in Eq. (48). It should be noted that the result will be identical to that of Eq. (16).

Performing similar operations as described in the above, since for $C_n \times C_m$ we have

$$M_{mn} = G_m(A_n, B_n, A_n) = I_m \otimes G_n(4,-1,4) + G_m(0,-1,0) \otimes I_n \qquad (56)$$

Therefore

$$\tau(C_n \times C_m) = \frac{n}{m}\prod_{i=2}^{m} det(G_n(4,-1,4) - 2\cos\frac{2i\pi}{m}I_n) \qquad (57)$$

Obviously the result will be identical to that of Eq. (17).

Now we study the strong Cartesian product of P_m and C_n. Her we have

$$M_{mn} = G_m(A_n, B_n, C_n) = I_m \otimes 3F_n(2,0,3) + G_m(-1,-1,-1) \otimes F_n(1,1,1) \qquad (58)$$

Since $\lambda_i(G_m(-1,-1,-1) = -(1+2\cos\frac{2i\pi}{m})$, therefore

$$eig(M_{mn}) = \bigcup_{i=1}^{m}\{ eig[3F_n(2,0,3) - (1+2\cos\frac{2i\pi}{m})F_n(1,1,1)]\} \qquad (59)$$

Therefore the matrix A_p will be

$$A_p = 3 * F_n(1,-1,2) \ ; \ \lambda_p = \lambda_i(A_p) = 2 + 2\cos\frac{i\pi}{n} \ ; \ i = 1:n \ ; \ \prod_{\substack{p=1 \\ \lambda_p \neq 0}}^{n} \lambda_p = n3^{n-1} \qquad (60)$$

Similarly for $G = C_n \boxtimes P_m$ we will have

$$\tau(G) = \frac{3^{n-1}}{m}\prod_{i=2}^{m} det[3F_n(2,0,2) - (1+2\cos\frac{2i\pi}{m})F_n(1,1,1)] \qquad (61)$$

Therefore if we transform a product into the form Eq. (52), then we can easily use Eq. (50) for calculating the number of spanning tree. As an example, suppose we want follow this process for a direct product. Consider the direct product of P_m and C_n. In this case we can write

$$M_{mn} = F_n(A_m, B_m, C_m) = I_n \otimes 2F_m(1,0,2) + G_n(0,-1,0) \otimes F_m(0,1,0) \tag{62}$$

Since we have $\lambda_i(G_n(0,-1,0) = -2\cos\dfrac{2i\pi}{n}$, therefore

$$eig(M_{mn}) = \bigcup_{i=1}^{n}\{eig[2F_m(1,0,2) - 2\cos\frac{2i\pi}{n}F_m(0,1,0)]\} \tag{63}$$

For the strong Cartesian product of C_m and C_n we will have

$$M_{mn} = G_n(A_m, B_m, A_m) = I_n \otimes G_m(8,-1,8) + G_n(0,-1,0) \otimes G_m(1,1,1) \tag{64}$$

Since $\lambda_i(G_n(0,-1,0) = -2\cos\dfrac{2i\pi}{n}$, therefore we have

$$eig(M_{mn}) = \bigcup_{i=1}^{n}\{eig[G_m(8,-1,8) - 2\cos\frac{2i\pi}{n}G_m(1,1,1)]\} \tag{65}$$

Thus one can easily obtain a relationship similar to that of Eq. (61). It is important to note that in all the calculation we arrive at the determinants of Eq. (44) or Eq. (48) which are previously calculated. Thus the operations will be confined to the multiplication of few parameters.

It should be noted that if A_p is different from what we have seen up to now, then the relationships of Ref. [14] can be employed.

Consider a matrix of the following form:

$$A_n = \begin{bmatrix} -\alpha+b & c & & & & \\ a & b & c & & & \\ & a & b & . & & \\ & & . & . & . & \\ & & & . & b & c \\ & & & & a & -\beta+b \end{bmatrix}_n \tag{66}$$

It is shown that the eigenvalue of the above matrix can be obtained as

$$ac\sin[(n+1)\theta] + (\alpha+\beta)\sqrt{ac}\sin(n\theta) + \alpha\beta\sin[(n-1)\theta] = 0 \tag{67}$$

$$\lambda = b + 2\sqrt{ac}\cos(\theta) \quad ; \quad \theta \neq m\pi \quad ; \quad m \in Z \tag{68}$$

6. CONCLUSION

In this paper using the two methods based on finding eigenvalues and determinant of the Laplacian matrices simple relationships are developed using the properties of the generators of the prduct graphs, The number of spanning trees of the product graphs are found for five well-documented graph products.

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REFERENCES

1. Kaveh A. *Optimal Structural Analysis Using The Cocept of Symmetry and Regularity*, Springer Verlag, GmbH, Wien - New York, 2013.
2. Bondy JA, Murty USR. *Graph Theory with Applications*, North-Holland, 1976.
3. Temperley H. On the mutual cancellation of cluster integrals in Mayer's fugacity series, *Proc. Phys. Soc.*, 1964; **83:** 3–16.
4. Kel'mans AK, Chelnokov VM. A certain polynomial of a graph and graphs with an extremal number of trees, *J. Comb. Theory Ser. B*, 1974; **16**: 197–214.
5. Prufer H. Neuer Beweis eines Stazes uber Permutationen. *Arch. Math. Phys.*, 1918; **27**: 742-744.
6. Kwaka JH, Parka YS, Satob I. Weighted complexities of graph products and bundles, *Euro. J. Comb.*, 2007; **28**: 228–245.
7. Sabidussi G. Graph Multiplication, *Math. Z.*, 1960; **72**: 446-457.
8. Weichsel PM. The Kronecker product of graphs, *Proc. Amer. Math. Soc*., 1962**; 13**: 47-52.
9. Harary F, Wilcox GW. Boolean operations on graphs, *Math. Scand*., 1967; **20**: 41-51.
10. Munarini E, Cippo CP, Scagliola A, Salvi NZ. Double graphs, *Dis. Math*., 2008; **308**: 242 – 254.
11. Kaveh A, Rahami H. Block Diagonalization of Adjacency and Laplacian Matrices for Graph Product; Applications in Structural Mechanics, *Int. J. Numer. Methods Eng*., 2006; **68**: 33-63.
12. Kaveh A, Rahami H. Compound Matrix Block Diagonalization for Efficient Solution of Eigenproblems in Structural Mechanics, *Acta Mech.*, 2007; **188**: 155-166.
13. Zhang F. *Matrix Theory. Basic Results and Techniques*, Springer-Verlag, New York, 1999.
14. Yueh WC. Eigenvalues of several tri-diagonal matrices. *Appl. Math*., 2005; **5**: 66–74.